The $U_q(\widehat{sl}(2|1))_1$-module $V(\Lambda_2)$ and a Corner Transfer Matrix at $q = 0$

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The north-west corner transfer matrix of an inhomogeneous integrable vertex model constructed from the vector representation of $U_q\big(sl(2|1)\big)$ and its dual is investigated. In the limit $q \to 0$, the spectrum can be obtained. Based on an analysis of the half-infinite tensor products related to all CTM-eigenvalues ≥ -4, it is argued that the eigenvectors of the corner transfer matrix are in one-to-one correspondence with the weight state of the $U_q\big(\widehat{sl}(2|1)\big)$-module $V(\Lambda_2)$ at level one. This is supported by a comparison of the complete set of eigenvectors with a nondegenerate triple of eigenvalues of the CTM-Hamiltonian and the generators of the Cartan-subalgebra of $U_q\big(sl(2|1)\big)$ to the weight states of $V(\Lambda_2)$ with multiplicity one.

Keywords: Integrable models, quantum affine superalgebras

I. INTRODUCTION

A variety of models with an underlying $gl(2|1)$-symmetry and their anisotropic generalizations with $U_q\big(gl(2|1)\big)$-invariance (for references see [1] and [2]) has been receiving attention following the introduction of the Perk-Schultz model [3] (or supersymmetric $t-J$ model [4]). The model specified in [3] is based on the R-matrix associated to the three-dimensional vector representation of $gl(2|1)$. Later, models constructed from other $gl(2|1)$- or $U_q\big(gl(2|1)$-representations have been investigated, among these the generalized supersymmetric Hubbard model related to the four-dimensional representation ([5] and references therein) or a family of "doped Heisenberg chains" involving atypical representations [20].

More recently, models incorporating various types of inhomogeneities have been considered. These include impurity $t-J$ models with the vector representation at one site (or few sites) of the quantum space substituted by a four-dimensional representation [7], [8] and periodic inhomogeneities. The latter may be implemented as a staggered disposition of the spectral parameter [9], [10] or both the spectral and the anisotropy parameter [2]. Another possibility consists in combining several representations in a periodic sequence. In [11], [12] the analysis of a model composed of the vector representation of $sl(2|1)$ and its dual is addressed to by means of the algebraic Bethe ansatz. Using the functional Bethe ansatz, the thermodynamic behavior of is extracted from the investigation of a model with the quantum space composed of an alternating sequence of these representations in [13].

This study is devoted to an integrable vertex model built from alternating sequences of the vector representation W of $U_q\big(sl(2|1)\big)$ and its dual W^* in both horizontal and vertical direction. In addition, the model allows for an inhomogeneity in the spectral parameters. Within this arrangement, the north-west corner transfer matrix (CTM) is analyzed in the limit $q \to 0$. Even though this limit does not exist for some elements of the R-matrix acting on $W \otimes W^*$ or $W^* \otimes W$, the elements of the composite R-matrix defined on $W \otimes W^* \otimes W \otimes W^*$ have a well defined limit. Suitable boundary conditions provided, corner transfer matrices map horizontal half-infinite sequences of vertical lattice links onto vertical half-infinite sequences of horizontal lattice links or vice versa. For vertex model based on quantum affine algebras, the corner transfer matrix is diagonal in the limit $q \to 0$ [23]. Though the structure of the corner transfer matrix elements remains nontrivial for the present model at $q = 0$, the simplification owed to this limit renders the spectrum accessible.

For a variety of integrable models associated to quantum algebras, the eigenvectors of the corner transfer matrices have been interpreted as weight vectors of level-k modules of the corresponding quantum affine algebra [14]- [19]. With these results, the concept of vertex operators [20] leads to a mathematical description of physical objects such as the transfer matrix or N-point correlators [21], [22]. These developments motivate the search for a similar identification of the CTM-eigenvectors of the mixed $U_q\big(sl(2|1)\big)$-model. To this aim, the eigenvectors attributed to each CTM-eigenvalue greater than -5 are compared to the weight states of the $U_q\big(\widehat{sl}(2|1)\big)$-module $V(\Lambda_2)$ at level one with grade greater than -5. The eigenvalues of the generators of the Cartan subalgebra acting on these eigenvectors via the infinitely folded coproduct are found in one-to-one correspondence to the weights at a given grade identified with the CTM-eigenvalue. Furthermore, all eigenvectors associated to nondegenerate CTM-eigenvalues are related to the

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weight states of $V(\Lambda_2)$ with multiplicity one. Relying on these results, a one-to-one correspondence between the CTM-eigenvectors and the weight states of the $U_q\big(\widehat{sl}(2|1)\big)_1$-module is conjectured. A more complete analysis as well as a description of the physical picture in terms of vertex operators will be published subsequently.

The paper is organized as follows. Section II recalls the definition of the quantum affine superalgebra $U_q\big(\widehat{sl}(2|1)\big)$ and the various R-matrices related to the vector representation of $U_q\big(sl(2|1)\big)$ and its dual. Section III specifies the integrable vertex model. In section IV, the Boltzmann weights of the elementary plaquettes composing the lattice model are evaluated in the limit $q \to 0$. The north-west corner transfer matrix for the inhomogeneous model is introduced in first part of section V. Its spectrum is investigated in subsections V B-V D. Section VI regards the module $V(\Lambda)$ and its relation to the half-infinite configurations on the lattice. Some details relevant to subsections V C and V D are provided in the appendix.

II. THE QUANTUM AFFINE SUPERALGEBRA $U_Q\big(\widehat{SL}(2|1)\big)$

The quantum affine superalgebra $U_q'\big(\widehat{sl}(2|1)\big)$ is the $\mathbf{C}$-algebra generated by $\{e_i, f_i, q^{\pm h_i},\ i = 0,1,2\}$ with the defining relations

$$\begin{aligned}
&q^{h_i}\, q^{h_j} = q^{h_j}\, q^{h_i} \\
&q^{h_i}\, e_j = q^{a_{ij}}\, e_j\, q^{h_i} \qquad\qquad q^{h_i}\, f_j = q^{-a_{ij}}\, f_j\, q^{h_i} \\
&[e_i, f_j] = \delta_{i,j}\, \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&f_0 f_0 f_i - [2] f_0 f_i f_0 + f_i f_0 f_0 = 0 \\
&e_0 e_0 e_i - [2] e_0 e_i e_0 + e_i e_0 e_0 = 0 \qquad \text{for } i = 1,2
\end{aligned} \tag{2}$$

$$\begin{aligned}
&[2]\big(f_1 f_2 f_1 f_0 f_2 + f_2 f_0 f_1 f_2 f_1 - f_2 f_1 f_2 f_0 f_1 - f_1 f_0 f_2 f_1 f_2 + f_1 f_2 f_0 f_2 f_1 - f_2 f_1 f_0 f_1 f_2\big) + \\
&\qquad + f_0 f_1 f_2 f_1 f_2 - f_1 f_2 f_1 f_2 f_0 - f_0 f_2 f_1 f_2 f_1 + f_2 f_1 f_2 f_1 f_0 = 0 \\
&[2]\big(e_1 e_2 e_1 e_0 e_2 + e_2 e_0 e_1 e_2 e_1 - e_2 e_1 e_2 e_0 e_1 - e_1 e_0 e_2 e_1 e_2 + e_1 e_2 e_0 e_2 e_1 - e_2 e_1 e_0 e_1 e_2\big) + \\
&\qquad + e_0 e_1 e_2 e_1 e_2 - e_1 e_2 e_1 e_2 e_0 - e_0 e_2 e_1 e_2 e_1 + e_2 e_1 e_2 e_1 e_0 = 0
\end{aligned} \tag{3}$$

The super commutator in the last equation of (1) is defined by

$$[a, b] = ab - (-1)^{|a|\cdot|b|}\, ba \qquad \forall a, b \in U_q'\big(\widehat{sl}(2|1)\big) \tag{4}$$

with the $\mathbf{Z}_2$-grading $|\cdot| : U_q'\big(\widehat{sl}(2|1)\big) \to \mathbf{Z}_2$ given by $|e_1| = |e_2| = |f_1| = |f_2| = 1$ and $|e_0| = |f_0| = |q^{h_i}| = 0$. Incorporating a generator d with the property

$$[d, e_1] = \delta_{i,0}\, e_i \qquad [d, f_i] = -\delta_{i,0}\, f_i \qquad [d, h_i] = 0 \qquad \text{for } i = 0,1,2 \tag{5}$$

yields the quantum affine superalgebra $U_q\big(\widehat{sl}(2|1)\big)$. Choosing both classical simple roots of the superalgebra odd, the matrix elements of the symmetrized Cartan matrix a are $a_{00} = 2,\ a_{11} = a_{22} = 0,\ a_{12} = a_{21} = -a_{01} = -a_{10} = -a_{02} = -a_{20} = 1$. In terms of the basis $\{\tau_1, \tau_2, \tau_3\}$ with the bilinear form $(\tau_i, \tau_j) = -(-1)^i \delta_{i,j}$, the classical simple roots $\bar\alpha_i$ and the classical weights $\bar\Lambda_i$ can be expressed by $\bar\alpha_i = -(-1)^i(\tau_i + \tau_{i+1})$ and $\bar\Lambda_i = \sum_{j=1}^{i} \tau_j - \delta_{i,1} \sum_{j=1}^{3} \tau_i$ with $i = 1,2$. An affine root δ and an affine weight Λ_0 are introduced by $(\Lambda_0, \Lambda_0) = (\delta, \delta) = (\Lambda_0, \tau_i) = (\delta, \tau_i) = 0$ and $(\Lambda_0, \delta) = 1$. Then the set of simple roots $\{\alpha_i\}_{i=0,1,2}$ of $U_q\big(\widehat{sl}(2|1)\big)$ is given by $\alpha_0 = \delta - \bar\alpha_1 - \bar\alpha_2$ and $\alpha_i = \bar\alpha_i$ for $i = 1,2$. The remaining affine weights are $\Lambda_i = \bar\Lambda_i + \Lambda_0$ with $i = 1,2$. The free abelian group $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\Lambda_2 \oplus \mathbf{Z}\delta$ is referred to as the weight lattice. Its dual lattice $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}h_2 \oplus \mathbf{Z}d$ may be identified with a subset of P via $(,)$ setting $\alpha_i = h_i$ and $d = \Lambda_0$.

A graded Hopf algebra structure is provided by the coproduct

$$\Delta(e_i) = q^{h_i} \otimes e_i + e_i \otimes 1 \qquad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i \qquad \Delta(q^{h_i}) = q^{h_i} \otimes q^{h_i} \tag{6}$$

the antipode

$$S(e_i) = -q^{-h_i}\, e_i \qquad S(f_i) = -f_i\, q^{h_i} \qquad S(q^{h_i}) = q^{-h_i} \tag{7}$$

and the counit

$$\epsilon(e_i) = \epsilon(f_i) = \epsilon(h_i) = 0 \qquad \epsilon(1) = 1 \tag{8}$$

The Drinfeld basis of generators proves well suited for purposes related to the bosonization of the superalgebra. In terms of this basis, $U'_q\big(\widehat{sl}(2|1)\big)$ is generated by $\{E^{i\pm}_m, H^i_n, q^{\pm h_i}\}$ with $i = 1, 2$, $m \in \mathbf{Z}, n \in \mathbf{Z} - \{0\}$ and the central elements $\gamma^{\pm\frac{1}{2}}$ subject to

$$\begin{aligned}
[H^i_n, H^j_m] &= \delta_{n+m,0}\, \frac{q^{a_{ij}n} - q^{-a_{ij}n}}{n(q-q^{-1})}\, \frac{\gamma^n - \gamma^{-n}}{q-q^{-1}} \\
q^{h_j}\, E^{i\pm}_m\, q^{-h_j} &= q^{\pm a_{ij}}\, E^{i\pm}_m \\
[H^i_n, E^{j\pm}_m] &= \pm \frac{q^{a_{ij}n} - q^{-a_{ij}n}}{n(q-q^{-1})}\, \gamma^{\mp\frac{|n|}{2}}\, E^{j\pm}_{n+m} \\
\Big[E^{i+}_n, E^{j-}_m\Big] &= \delta_{ij}\, \frac{\gamma^{\frac{n-m}{2}}\, \Psi^{i+}_{n+m} - \gamma^{\frac{m-n}{2}}\, \Psi^{i-}_{n+m}}{q-q^{-1}}
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
\sum_{n\geq 0} \Psi^{i+}_n z^{-n} &= q^{h_i}\, \exp\Big((q-q^{-1}) \sum_{n>0} H^i_n z^{-n}\Big) \\
\sum_{n\geq 0} \Psi^{i-}_{-n} z^{n} &= q^{-h_i}\, \exp\Big(-(q-q^{-1}) \sum_{n>0} H^i_{-n} z^{n}\Big)
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\Big[E^{i\pm}_m, E^{i\pm}_n\Big] &= 0 \qquad i = 1, 2 \\
E^{1\pm}_{n+1} E^{2\pm}_m + q^{\pm 1}\, E^{2\pm}_m E^{1\pm}_{n+1} &= q^{\pm 1}\, E^{1\pm}_n E^{2\pm}_{m+1} + E^{2\pm}_{m+1} E^{1\pm}_n
\end{aligned} \tag{11}$$

The above choice of simple roots implies the $\mathbf{Z}_2$-grading $|E^{i\pm}_m| = 1$ and $|H^i_n| = |q^{\pm h_i}| = |\gamma| = 0$. The Drinfeld generators are related to the Chevalley basis (1) -(3) by

$$\begin{aligned}
e_i &= E^{i+}_0 \qquad f_i = E^{i-}_0 \qquad \text{for } i = 1, 2 \\
e_0 &= \Big(E^{2-}_0 E^{1-}_1 + q\, E^{1-}_1 E^{2-}_0\Big)\, q^{-h_1-h_2} \\
f_0 &= -q^{h_1+h_2}\, \Big(E^{1+}_{-1} E^{2+}_0 + q^{-1}\, E^{2+}_0 E^{1+}_{-1}\Big)
\end{aligned} \tag{12}$$

A three-dimensional module W of $U'_q\big(\widehat{sl}(2|1)\big)$ with basis $\big\{w_i\big\}_{0\leq i\leq 2}$ is given by

$$\begin{aligned}
f_0\, w_2 &= q w_0 \qquad & e_0\, w_0 &= q^{-1} w_2 \\
f_1\, w_0 &= w_1 \qquad & e_1\, w_1 &= w_0 \\
f_2\, w_1 &= w_2 \qquad & e_2\, w_2 &= -w_1
\end{aligned} \tag{13}$$

and

$$
\begin{aligned}
h_0\, w_0 &= -w_0 & h_1\, w_0 &= w_0 & h_2\, w_0 &= 0 \\
h_0\, w_1 &= 0 & h_1\, w_1 &= w_1 & h_2\, w_1 &= -w_1 \\
h_0\, w_2 &= w_2 & h_1\, w_2 &= 0 & h_2\, w_2 &= -w_2
\end{aligned} \tag{14}
$$

The $\mathbf{Z}_2$-grading on W is fixed by $|w_0| = |w_2| = 0$ and $|w_1| = 1$. Given an anti-automorphism ϕ of the superalgebra, the dual space of W endowed with $U_q'\big(\widehat{sl}(2|1)\big)$-structure

$$
\langle x w^* | w \rangle = (-1)^{|x|\cdot|w^*|}\, \langle w^* | \phi(x) w \rangle \qquad \forall x \in U_q'\big(\widehat{sl}(2|1)\big) \tag{15}
$$

is usually denoted by $W^{*\phi}$. In the following, W^{*S} will be denoted by W^* for brevity. The basis $\big\{w_i^*\big\}_{0\le i\le 2}$ of W^* with $|w_0^*| = |w_2^*| = 0$ and $|w_1^*| = 1$ may be chosen such that the action of the superalgebra reads

$$
\begin{aligned}
f_1\, w_1^* &= q w_0^* & e_1\, w_0^* &= -q^{-1} w_1^* \\
f_2\, w_2^* &= -q^{-1} w_1^* & e_2\, w_1^* &= -q w_2^* \\
f_0\, w_0^* &= -q^2 w_2^* & e_0\, w_2^* &= -q^{-2} w_0^*
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
h_0\, w_0^* &= w_0^* & h_1\, w_0^* &= -w_0^* & h_2\, w_0^* &= 0 \\
h_0\, w_1^* &= 0 & h_1\, w_1^* &= -w_1^* & h_2\, w_1^* &= w_1^* \\
h_0\, w_2^* &= -w_2^* & h_1\, w_2^* &= 0 & h_2\, w_2^* &= w_2^*
\end{aligned} \tag{17}
$$

A $U_q\big(\widehat{sl}(2|1)\big)$-structure can be implemented on the evaluation modules $W_z = W \otimes F[z, z^{-1}]$ and $W_z^* = W^* \otimes F[z, z^{-1}]$ via

$$
\begin{aligned}
e_i\big(v_j \otimes z^n\big) &= e_i v_j \otimes z^{n+\delta_{i,0}} & f_i\big(v_j \otimes z^n\big) &= f_i v_j \otimes z^{n-\delta_{i,0}} \\
h_i\big(v_j \otimes z^n\big) &= h_i v_j \otimes z^n & d\big(v_j \otimes z^n\big) &= n\, v_j \otimes z^n
\end{aligned} \tag{18}
$$

with $i, j = 0, 1, 2$ and $v_j = w_j$ or $v_j = w_j^*$.

For two evaluation modules $V_{z_m}^{(m)} = W_{z_m}$ or $V_{z_m}^{(m)} = W_{z_m}^*$, the R-matrix $R(z_1/z_2) \in End\big(V_{z_1}^{(1)} \otimes V_{z_2}^{(2)}\big)$ intertwines the action of $U_q\big(\widehat{sl}(2|1)\big)$ according to

$$
R(z_1/z_2)\, \Delta(x) = \Delta'(x)\, R(z_1/z_2) \qquad \forall x \in U_q\big(\widehat{sl}(2|1)\big) \tag{19}
$$

where $\Delta' = \sigma \circ \Delta$ and $\sigma(x \otimes y) = (-1)^{|x|\cdot|y|}\, y \otimes x$. In the remainder, subscripts will indicate the choices of the evaluation modules where required for clarity. The corresponding R-matrix elements are introduced by

$$
\begin{aligned}
R_{WW}(z_1/z_2)\big(w_i \otimes w_j\big) &= \sum_{k,l=0,1,2} R_{i,j}^{k,l}(z_1/z_2)\, w_k \otimes w_l \\
R_{W^*W^*}(z_1/z_2)\big(w_i^* \otimes w_j^*\big) &= \sum_{k,l=0,1,2} R_{i^*,j^*}^{k^*,l^*}(z_1/z_2)\, w_k^* \otimes w_l^* \\
R_{WW^*}(z_1/z_2)\big(w_i \otimes w_j^*\big) &= \sum_{k,l=0,1,2} R_{i,j^*}^{k,l^*}(z_1/z_2)\, w_k \otimes w_l^* \\
R_{W^*W}(z_1/z_2)\big(w_i^* \otimes w_j\big) &= \sum_{k,l=0,1,2} R_{i^*,j}^{k^*,l}(z_1/z_2)\, w_k^* \otimes w_l
\end{aligned} \tag{20}
$$

Up to a scalar multiple, the R-matrix elements (20) are uniquely determined by the intertwining property (19). For $V_{z_1}^{(1)} = W_{z_1}$ and $V_{z_2}^{(2)} = W_{z_2}$, a solution of (19) is given by

$$
\begin{aligned}
&R^{0,0}_{0,0}(z)=R^{2,2}_{2,2}(z)=1 \qquad\qquad R^{1,1}_{1,1}(z)=\frac{q^2-z}{1-q^2z}\\
&R^{i,j}_{i,j}(z)=\frac{q(1-z)}{1-q^2z} \qquad i\neq j\\
&R^{j,i}_{i,j}(z)=-\frac{(q^2-1)z}{1-q^2z} \qquad i<j\\
&R^{j,i}_{i,j}(z)=-\frac{q^2-1}{1-q^2z} \qquad i>j
\end{aligned}
\tag{21}
$$

The solution satisfies the initial condition

$$
R^{k,l}_{i,j}(1)=R^{k^*,l^*}_{i^*,j^*}(1)=\delta_{i,l}\delta_{j,k}\,(-1)^{|k|\cdot|l|}
\tag{22}
$$

Relations between R-matrix elements with respect to the various choices of evaluation modules are conveniently stated introducing

$$
\begin{aligned}
\bar{R}^{k,l}_{i,j}(z) &= (-1)^{|k|\cdot|l|}\,R^{k,l}_{i,j}(z) & \bar{R}^{k^*,l^*}_{i^*,j^*}(z) &= (-1)^{|k|\cdot|l|}\,R^{k^*,l^*}_{i^*,j^*}(z)\\
\bar{R}^{k^*,l}_{i^*,j}(z) &= (-1)^{|k|\cdot|l|}\,R^{k^*,l}_{i^*,j}(z) & \bar{R}^{k,l^*}_{i,j^*}(z) &= (-1)^{|k|\cdot|l|}\,R^{k,l^*}_{i,j^*}(z)
\end{aligned}
\tag{23}
$$

Then solutions to the intertwining condition (19) are obtained from (21), (23) and

$$
\begin{aligned}
&\bar{R}^{k^*,l^*}_{i^*,j^*}(z)=\bar{R}^{i,j}_{k,l}(z)\\
&\bar{R}^{k,l^*}_{i,j^*}(z)=(-1)^{|k|-|i|}\;\bar{R}^{j,k}_{l,i}\big(1/q^2z\big)\\
&\bar{R}^{k^*,l}_{i^*,j}(z)=\bar{R}^{l,i}_{j,k}\big(1/z\big)
\end{aligned}
\tag{24}
$$

In terms of the R-matrix elements (20), the second inversion relation reads

$$
R^{k,l}_{i,j}(z)=R^{j,i}_{l,k}(z) \qquad R^{k^*,l^*}_{i^*,j^*}(z)=R^{j^*,i^*}_{l^*,k^*}(z) \qquad R^{k^*,l}_{i^*,j}(z)=R^{j,i^*}_{l,k^*}\big(z/q^2\big)
\tag{25}
$$

A further symmetry of the R-matrix is stated by

$$
\begin{aligned}
\bar{R}^{k,l}_{i,j}(z) &= z^{\frac{1}{2}(k+\delta_{k,1}-i-\delta_{i,1})}\;\bar{R}^{l,k}_{j,i}(z) & \bar{R}^{k^*,l^*}_{i^*,j^*}(z) &= z^{-\frac{1}{2}(k+\delta_{k,1}-i-\delta_{i,1})}\;\bar{R}^{l^*,k^*}_{j^*,i^*}(z)\\
\bar{R}^{k,l^*}_{i,j^*}(z) &= \big(q^2z\big)^{\frac{1}{2}(k+\delta_{k,1}-i-\delta_{i,1})}\;\bar{R}^{l,k^*}_{j,i^*}(z) & \bar{R}^{k^*,l}_{i^*,j}(z) &= z^{-\frac{1}{2}(k+\delta_{k,1}-i-\delta_{i,1})}\;\bar{R}^{l^*,k}_{j^*,i}(z)
\end{aligned}
\tag{26}
$$

III. THE LATTICE MODEL

A section of the infinite lattice model considered in the following sections is illustrated in Fig. 1. The modules W and W^* are associated in an alternating sequence to the horizontal and vertical lines as indicated by arrows pointing right or upwards for W and pointing left or downwards for W^*. Hence, the lattice model may be decomposed into elementary plaquettes shown in Fig. 2. For each vertex, the matrix elements (23) provide Boltzmann weights depending on the configuration of basis elements $\{w_i\}_{0\leq i\leq 2}$ or $\{w^*_i\}_{0\leq i\leq 2}$ attributed to the adjacent links and on the spectral parameters chosen for the type of vertex as specified in the left part of Fig. 2.

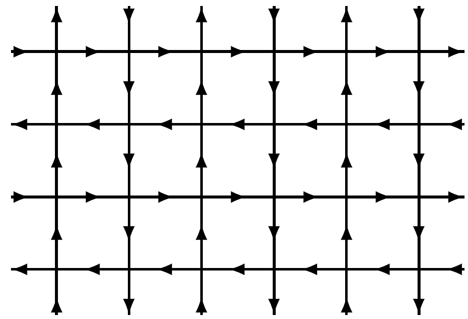
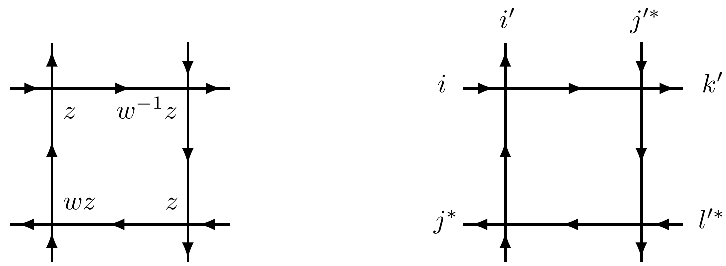


Fig. 1: The lattice model



Spectral parameters Assignment of indices

Fig. 2: The elementary plaquette

The present investigation applies to any finite value w specifying the inhomogeneity in the spectral parameters. An analysis of the limits $w \to 0$ and $w \to \infty$, where the R-matrices on $W \otimes W^*$ and $W^* \otimes W$ tend towards their braid limits, will be presented separately.

With the assignment of indices shown in the right part of Fig. 2, the Boltzmann weight corresponding to an elementary plaquette is given by

$$R^{i',j'^*;k',l'^*}_{i,\,j^*;\,k,\,l^*}(w,z) = \sum_{\tilde{j},\tilde{k},\tilde{l},\tilde{n}=0,1,2} \bar{R}^{\,k',j'^*}_{\,\tilde{n},\,\tilde{l}^*}(w^{-1}z)\; \bar{R}^{\,l'^*,\tilde{l}^*}_{\,\tilde{j}^*,l^*}(z)\; \bar{R}^{\,\tilde{n},i'}_{\,i,\tilde{k}}(z)\; \bar{R}^{\,\tilde{j}^*,\tilde{k}}_{\,j^*,k}(wz) \tag{27}$$

Use of the solutions (21)-(24) of the intertwining condition (19) for the matrix elements in the last equation yields an integrable vertex model. Due to the initial condition (22) and the unitarity relation $\sum_{k,l=0,1,2} \bar{R}^{j',i'^*}_{l,\,k^*}(w^{-1})\bar{R}^{k^*,l}_{i^*,j}(w) = \delta_{i,i'}\delta_{j,j'}$ the matrix elements of $R(w,z) : \left(\otimes W \otimes W^*\right)^2 \longrightarrow \left(\otimes W \otimes W^*\right)^2$ satisfy

$$R^{i',j'^*;k',l'^*}_{i,\,j^*;\,k,\,l^*}(w,1) = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}\delta_{l,l'} \tag{28}$$

The property (28) may be viewed as initial condition for $R(w,z)$.

IV. THE LIMIT $Q \to 0$

The Boltzmann weights of homogeneous vertex models based on finite-dimensional representations of quantum algebras simplify drastically in the limit $q \to 0$ [23]. A remarkable simplification occurs in the present case, too. Well-defined limits are found for the Boltzmann weights (27) of the elementary plaquettes even though this is not the case for all single R-matrix elements (24). Keeping z and w fixed, the matrix elements $R^{i',j'^*;k',l'^*}_{i,\,j^*;\,k,\,l^*}(qw,z)$ tend to values independent of w when the limit $q \to 0$ is performed:

$$\lim_{q\to 0} R^{i',j'^*;k',l'^*}_{i,\,j^*;\,k,\,l^*}(qw,z) \equiv P^{i',j'^*;k',l'^*}_{i,\,j^*;\,k,\,l^*}(z) \tag{29}$$

Use of the explicit expressions (21) in (24) yields the following results for these limits :

$$P^{i,j^*;k,l^*}_{i,j^*;k,l^*}(z) = z^{-y_{i,j,k}-y_{l,k,j}} \tag{30}$$

with

$$\begin{aligned}
y_{i,j,k} &= 0 \qquad && i<k \\
y_{i,j,k} &= 1 \qquad && i>k,\ j\neq 0 \text{ for } i=1 \\
y_{1,0,0} &= 0 \qquad && \\
y_{i,j,k} &= 0 \qquad && i=k=1 \\
y_{i,j,k} &= 1 \qquad && i=k=0,2, \quad j\neq i \\
y_{i,j,k} &= 0 \qquad && i=j=k=0,2
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
&P^{i,(1^*;1+0^*;0),l^*}_{i,(1^*;1-0^*;0),l^*}(z) = 0 \\
&P^{i,(1^*;1-0^*;0),l^*}_{i,(1^*;1+0^*;0),l^*}(z) = 2(\delta_{i,1}+\delta_{l,1})\cdot \frac{z-1}{z^{1+\delta_{i,2}+\delta_{l,2}}} \qquad \forall i,l
\end{aligned} \tag{32}$$

In addition, the following nondiagonal matrix elements of $P(z)$ are found:

$$\begin{aligned}
P^{k,1^*;1,l^*}_{1,1^*;k,l^*}(z) &= -P^{k,0^*;0,l^*}_{1,1^*;k,l^*}(z) = \frac{z-1}{z^{1+\delta_{l,2}}} \qquad \text{for } k\neq 1 \\
P^{i,1^*;1,j^*}_{i,j^*;1,1^*}(z) &= -P^{i,0^*;0,j^*}_{i,j^*;1,1^*}(z) = \frac{z-1}{z^{1+\delta_{i,2}}} \qquad \text{for } j\neq 1
\end{aligned} \tag{33}$$

$$P^{k,1^*;1,l^*}_{2,2^*;k,l^*}(z) = -P^{k,0^*;0,l^*}_{2,2^*;k,l^*}(z) = -\frac{z^{1+\delta_{k,2}\delta_{l,2}}-1}{z^{1+\delta_{l,2}}} \qquad P^{i,1^*;1,j^*}_{i,j^*;2,2^*}(z) = -P^{i,0^*;0,j^*}_{i,j^*;2,2^*}(z) = -\frac{z^{1+\delta_{i,2}\delta_{j,2}}-1}{z^{1+\delta_{i,2}}} \tag{34}$$

and

$$\begin{aligned}
P^{1,1^*;i,l^*}_{i,1^*;1,l^*}(z) + P^{1,1^*;i,l^*}_{i,0^*;0,l^*}(z) &= 2\,\frac{z-1}{z^{1+\delta_{l,2}}} \qquad i\neq 1 \\
P^{i,l^*;1,1^*}_{i,1^*;1,l^*}(z) + P^{i,l^*;1,1^*}_{i,0^*;0,l^*}(z) &= 2\,\frac{z-1}{z^{1+\delta_{i,2}}} \qquad l\neq 1 \\
P^{2,2^*;i,l^*}_{i,1^*;1,l^*}(z) + P^{2,2^*;i,l^*}_{i,0^*;0,l^*}(z) &= 2\,\frac{z^{1+\delta_{i,2}\delta_{l,2}}-1}{z^{1+\delta_{l,2}}} \\
P^{i,l^*;2,2^*}_{i,1^*;1,l^*}(z) + P^{i,l^*;2,2^*}_{i,0^*;0,l^*}(z) &= 2\,\frac{z^{1+\delta_{i,2}\delta_{l,2}}-1}{z^{1+\delta_{i,2}}}
\end{aligned} \tag{35}$$

All other matrix elements of $P(z)$ vanish. In contrast to the cases studied in [23], the matrix of Boltzmann weights does not assume a diagonal form in the limit of vanishing q. However, the simplification proves sufficient to establish a link to the representation theory of the affine superalgebra. As in the case of lattice models related to finite-dimensional representations of quantum affine algebras [14]- [19], this link is provided by the corner transfer matrices of the model.

V. THE CORNER TRANSFER MATRIX IN THE LIMIT $Q \to 0$

A. The inhomogeneous corner transfer matrix

Corner transfer matrices may be introduced for the present model in close analogy to the construction developed for the eight-vertex model in [24,25]. A triangular subsection A_N of the upper left quadrant is considered. Its vertical(horizontal) boundaries coincide with $2N+1$ horizontal(vertical) links on the boundaries of the quadrant. For a fixed configuration of basis elements $\{w_i\}_{0\leq i\leq 2}$ or $\{w_i^*\}_{0\leq i\leq 2}$ on all links of its diagonal boundary, the Boltzmann weights of the subsection A_N yield a map

$$A^{(N)}(w,z):\ W^*\big(\otimes W\otimes W^*\big)^{N+1} \longrightarrow W^*\big(\otimes W\otimes W^*\big)^{N+1} \tag{36}$$

At $z=1$, (36) reduces to the identity map due to the initial conditions (22) and (28). Thus a Hamiltonian corresponding to the section A_N can be introduced by

$$h^{(N)}_{CTM}(w) \equiv (N+1)h_{2N+2,2N+1} + \sum_{\tilde N=1}^{N} \tilde N\, h_{2(\tilde N+1),2\tilde N+1;\ 2\tilde N,2\tilde N-1}(w) \tag{37}$$

In (37), $h_{2(\tilde N+1),2\tilde N+1;\ 2\tilde N,2\tilde N-1}(w)$ denotes the operator $h(w)$ acting on the $(2\tilde N-1)$-th to the $(2(\tilde N+1))$-th component of $\big(\otimes W\otimes W^*\big)^{N+1}$ counted from the right, with $h(w)$ defined by the expansion

$$R(w,z) = \mathbf{1} + (z-1)\,h(w) + O\big((z-1)^2\big) \tag{38}$$

Similarly, $h_{2N+2,2N+1}$ is the operator h defined by $\sigma\, R_{W^*W^*}(z) = \mathbf{1}+(z-1)\,h+O\big((z-1)^2\big)$ acting on the two leftmost components of $W^*\big(\otimes W\otimes W^*\big)^{N+1}$. The boundary condition imposed on the diagonal boundary of A_N has to be chosen consistent with the expansion of $R(w,z)$ and $\sigma R_{W^*W^*}(z)$ at $z=1$. With respect to a particular boundary condition, the large-N limit of $h^{(N)}_{CTM}(w)$ is referred to as the corner transfer matrix Hamiltonian $h_{CTM}(w)$. The boundary condition adopted in the following is specified by attributing only w_2 or w_2^* to any link on the diagonal boundary. In the remainder of this section, $h^{(N)}_{CTM}(w)$ will be investigated in the limit $q\to 0$ and $N\to\infty$. The corresponding operator is denoted by

$$H_0 = \lim_{N\to\infty}\lim_{q\to 0} h^{(N)}_{CTM}(w) \tag{39}$$

Examination of (30)-(35) shows that H_0 does not depend on the (finite) value of w.

B. A restricted set of configurations

A particular configuration $\big(\ldots\otimes w_{i_2}\otimes w^*_{j_2}\otimes w_{i_1}\otimes w^*_{j_1}\big)$ on the horizontal or vertical boundary of the corner transfer matrix may be abbreviated writing

$$\big(\ldots, j_2^* i_2, j_1^* i_1, j_0^*\big) \equiv \big(\ldots\otimes w^*_{j_2}\otimes w_{i_2}\otimes w^*_{j_1}\otimes w_{i_1}\otimes w^*_{j_0}\big) \tag{40}$$

According to the boundary condition fixed above, only finitely many i_n, j_n differ from 2. For a large subset of configurations (40) to be specified in this subsection, the matrix elements of the corner transfer Hamiltonian can be arranged in a trigonal form.

Making use of (30) and (34) in (37)-(39) and of (21)-(24), the action of H_0 on the configuration $\big(\ldots,2^*2,2^*2,i^*\big)$ with $i=0,1,2$ is easily evaluated:

$$\begin{aligned} H_0\big(\ldots,2^*2,2^*2,i^*\big) =&\ -(2-\delta_{i,0}\delta_{i,1})\big(\ldots,2^*2,2^*2,(1^*1-0^*0),i^*\big)\\ &-2\Big\{2\big(\ldots,2^*2,2^*2,(1^*1-0^*0),2^*2,i^*\big)+3\big(\ldots,2^*2,(2^*2,1^*1-0^*0),2^*2,2^*2,i^*\big)+\ldots\Big\} \end{aligned} \tag{41}$$

The configurations on the rhs of (41) are obtained from $\big(\ldots,2^*2,2^*2,i^*\big)$ by replacing one subsequence 2^*2 by the difference 1^*1-0^*0. Taking into account (30), (31) and (32), the action of H_0 on these configurations is found:

$$H_0\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^n,i^*\big)=-\big(2(n+1)-\delta_{n,0}(\delta_{i,0}+\delta_{i,1})\big)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^n,i^*\big)$$

$$-\sum_{m=0}^{n-2}(m+1)(2-\delta_{m,0}\,\delta_{i,0}\delta_{i,1})\,\big(\ldots,2^*2,2^*2,\big(1^*1-0^*0),(2^*2)^{n-m-1},(1^*1-0^*0),(2^*2)^m,i^*\big)$$

$$-(n-\delta_{n,1}\,\delta_{i,0}\delta_{i,1})\big(\ldots,2^*2,2^*2,(1^*1-0^*0)^2,(2^*2)^{n-1},i^*\big)$$

$$-(n+2)\big(\ldots,2^*2,2^*2,(1^*1-0^*0)^2,(2^*2)^n,i^*\big)$$

$$-2\Big\{(n+3)\big(\ldots,2^*2,2^*2,(1^*1-0^*0),2^*2,(1^*1-0^*0),(2^*2)^n,i^*\big)+$$

$$+(n+4)\big(\ldots,2^*2,2^*2,(1^*1-0^*0,)2^*2,2^*2,(1^*1-0^*0),(2^*2)^n,i^*\big)+$$

$$+(n+5)\big(\ldots,2^*2,2^*2,(1^*1-0^*0),2^*2,2^*2,2^*2,(1^*1-0^*0),(2^*2)^n,i^*\big)+\ldots\Big\} \qquad (42)$$

With the second line dropped for $n=0,1$, equation (42) is valid for all $n\geq 0$. Inspection of (30)-(35) allows for a description of the repeated action of H_0 on the configurations on the rhs of (41), (42). To facilitate notation of the explicit expressions it is useful to introduce the abbreviation

$$\tau\big\{k^{(l)}_{t_l}\big\}_{1\leq t_l\leq s_l}\equiv,(1^*1-0^*0)^{k_1^{(l)}},2^*2,(1^*1-0^*0)^{k_2^{(l)}},2^*2,(1^*1-0^*0)^{k_3^{(l)}},\ldots,2^*2,(1^*1-0^*0)^{k_{s_l}^{(l)}}, \qquad (43)$$

with $s_l=1,2,3,\ldots$ and $k^{(l)}_{t_l}=1,2,3,\ldots$ for $1\leq t\leq s_l$. Any configuration emerging from repeated action of h_0 on $\big(\ldots,2^*2,2^*2,i^*\big)$ can be written

$$\Big(\ldots,2^*2,2^*2\,\tau\big\{k^{(R)}_{t_R}\big\}_{1\leq t_R\leq s_R}(2^*2)^{n_R}\tau\big\{k^{(R-1)}_{t_{R-1}}\big\}_{1\leq t_{R-1}\leq s_{R-1}}(2^*2)^{n_{R-1}},\ldots,(2^*2)^{n_2}\tau\big\{k^{(1)}_{t_1}\big\}_{1\leq t_1\leq s_1}(2^*2)^{n_1},i^*\Big) \qquad (44)$$

with $n_1\geq 0$ and $n_l\geq 2$ for $l>1$. A configuration of the form (44) is composed from $r=\sum_{l=1}^R\sum_{t_l=1}^{s_l}k^{(l)}_{t_l}$ subsequences 1^*1-0^*0 placed between subsequences 2^*2 and the right end $\ldots,i^*\big)$. The action of H_0 on (44) is given by

$$H_0\bigg(\Big(\ldots,2^*2,\;\tau\big\{k^{(R)}_{t_R}\big\}_{1\leq t_R\leq s_R}(2^*2)^{n_R}\tau\big\{k^{(R-1)}_{t_{R-1}}\big\}_{1\leq t_{R-1}\leq s_{R-1}}(2^*2)^{n_{R-1}},\ldots,(2^*2)^{n_2}\tau\big\{k^{(1)}_{t_1}\big\}_{1\leq t_1\leq s_1}(2^*2)^{n_1},i^*\Big)\bigg)$$

$$=-\alpha\big(\{k^{(l)}_{t_l}\}_{1\leq t_l\leq s_l,\,1\leq l\leq R},\;\{n_l\}_{1\leq l\leq R}\big)\cdot$$

$$\Big(\ldots,2^*2,\;\tau\big\{k^{(R)}_{t_R}\big\}_{1\leq t_R\leq s_R}(2^*2)^{n_R}\tau\big\{k^{(R-1)}_{t_{R-1}}\big\}_{1\leq t_{R-1}\leq s_{R-1}}(2^*2)^{n_{R-1}},\ldots,(2^*2)^{n_2}\tau\big\{k^{(1)}_{t_1}\big\}_{1\leq t_1\leq s_1}(2^*2)^{n_1},i^*\Big)$$

$$-\sum_{m_1=0}^{n_1-1}(2-\delta_{m_1,n_1-1}-\delta_{m_1,0}\,\delta_{i,0}\delta_{i,1})\,(m_1+1)\cdot$$

$$\Big(\ldots,2^*2\,\tau\big\{k^{(R)}_{t_R}\big\}_{1\leq t_R\leq s_R}(2^*2)^{n_R},\ldots,(2^*2)^{n_2}\tau\big\{k^{(1)}_{t_1}\big\}_{1\leq t_1\leq s_1}(2^*2)^{n_1-m_1-1},(1^*1-0^*0),(2^*2)^{m_1}i^*\Big)$$

$$-\sum_{S=2}^{R}\sum_{m_S=0}^{n_S-1}(2-\delta_{m_S,0}-\delta_{m_S,n_S-1})\left(\sum_{l=1}^{S-1}(k_1^{(l)}+k_2^{(l)}+\ldots+k_{s_l}^{(l)}+s_l+n_l-1)+m_S+1\right)\cdot$$

$$\Big(\ldots,2^*2\,\tau\big\{k^{(R)}_{t_R}\big\}_{1\leq t_R\leq s_R}(2^*2)^{n_R},\ldots,(2^*2)^{n_{S+1}}\tau\big\{k^{(S)}_{t_S}\big\}_{1\leq t_S\leq s_S}(2^*2)^{n_S-m_S-1},(1^*1-0^*0),$$

$$(2^*2)^{m_S}\tau\big\{k^{(S-1)}_{t_{S-1}}\big\}_{1\leq t_{S-1}\leq s_{S-1}}(2^*2)^{n_{S-1}},\ldots,(2^*2)^{n_2}\tau\big\{k^{(1)}_{t_1}\big\}_{1\leq t_1\leq s_1}(2^*2)^{n_1}i^*\Big)$$

$$-\sum_{m=0}^{\infty}(2-\delta_{m,0})\Big(\sum_{l=1}^{R}\big(k_1^{(l)}+k_2^{(l)}+\ldots+k_{s_l}^{(l)}+s_l+n_l-1\big)+m+1\Big)\cdot$$

$$\Big(\ldots, 2^*2, (1^*1-0^*0), (2^*2)^m\, \tau\big\{k^{(R)}_{t_R}\big\}_{1\le t_R\le s_R}(2^*2)^{n_R}, \ldots, (2^*2)^{n_2}\tau\big\{k^{(1)}_{t_1}\big\}_{1\le t_1\le s_1}(2^*2)^{n_1}, i^*\Big) \tag{45}$$

with

$$\begin{aligned}
\alpha\big(\{k^{(l)}_{t_l}\}_{1\le t_l\le s_l,\, 1\le l\le R},\ \{n_l\}_{1\le l\le R}\big) &= \sum_{l=1}^{R}\big(s_l^2 + (2s_l-1)k^{(l)}_{s_l} + (2s_l-3)k^{(l)}_{s_l-1} + \ldots + 3k^{(l)}_2 + k^{(l)}_1\big)\\
&\quad + 2\sum_{l=2}^{R} s_l\Big(\sum_{m=1}^{l-1}\big(k^{(m)}_1 + k^{(m)}_2 + \ldots + k^{(m)}_{s_m} + s_m + n_{m+1} - 1\big)\Big)\\
&\quad - \delta_{n_1,0}\,\delta_{i,0} - \delta_{n_1,0}\,\delta_{i,1} + 2n_1\sum_{l=1}^{R} s_l
\end{aligned} \tag{46}$$

The fourth and fifth line of (45) give no contribution for $n_1 = 0$. The configuration in the first term on the rhs of (45) coincides with the configuration on the lhs. Obviously, all other configurations on the rhs can be rewritten as configurations of the type (44) built from $r+1$ subsequences $1^*1 - 0^*0$ between subsequences 2^*2 and the right border $\ldots, i^*\big)$. Hence, for each $i = 0, 1, 2$ the set of all configurations of the form (44) with $r = 0, 1, 2, \ldots$ is closed under the action of H_0. For a given i, the corresponding matrix elements of H_0 form a triangular matrix with the diagonal matrix elements given by (46). Since these are all smaller than zero for a configuration different from $\big(\ldots, 2^*2, 2^*2, i^*\big)$, an eigenvector of H_0 with eigenvalue zero is given by a linear combination of all configurations (44) for each fixed i. An eigenvector of H_0 with eigenvalue $-2(n+1)$ is provided by a linear combination involving $\big(\ldots, 2^*2, 2^*2, (1^*1-0^*0), (2^*2)^n, 2^*\big)$ and all configurations generated from these by repeated action of H_0. Among the latter, the configurations $\big(\ldots, 2^*2, 2^*2, (1^*1-0^*0)^{1+2m}, (2^*2)^{n-m}, 2^*\big)$ with $1 \le m \le n$ have the same value of the diagonal element (46) as $\big(\ldots, 2^*2, 2^*2, (1^*1-0^*0), (2^*2)^n, 2^*\big)$. It is easily verified that these don't occur in the appropriate linear combination. Analogously, an eigenvector of H_0 can be constructed as a linear combination of any configuration (44) and all configurations obtained from it by repeated action of H_0 with a different diagonal element (46). The corresponding eigenvalue coincides with the diagonal element (46) of the particular configuration chosen. In this context, diagonalizability is a consequence of the particular form of the corner transfer matrix Hamiltonian (37), (39) and does not apply to the $q \to 0$-limit of the transfer matrix Hamiltonian acting on the configurations (44), for example.

A similar structure applies to configurations $\big(\ldots, 2^*2, 2^*2, 2^*(0, 2^*)^l(2, 2^*)^n\big)$ with $n = 0, 1, 2, \ldots$ and $l = 1, 2, 3, \ldots$. Given fixed values of n, l, the collection of states

$$\begin{aligned}
&\big(\ldots, 2^*2, 2^*2, 2^*(0, 2^*)^l(2, 2^*)^n\big)\\
&\big(\ldots, 2^*2, 2^*2, 2^*0, (1^*1-0^*0), 2^*(0, 2^*)^{l-1}(2, 2^*)^n\big)\\
&\big(\ldots, 2^*2, 2^*2, 2^*(0, 2^*)^{l-1}0, (1^*1-0^*0), 2^*(2, 2^*)^{n-1}\big) \qquad \text{for } n > 0\\
&\big(\ldots, 2^*2, 2^*2, 2^*0, (1^*1-0^*0), 2^*(0, 2^*)^{l-2}0, (1^*1-0^*0), 2^*(2, 2^*)^{n-1}\big) \qquad \text{for } n > 0, l > 1
\end{aligned} \tag{47}$$

and all configurations obtained from these by replacing one or more subsequences 2^*2 by $1^*1 - 0^*0$ is closed under the action of H_0. Application of H_0 on one of these configurations with r subsequences $1^*1 - 0^*0$ yields the same configuration with a prefactor depending on the position of these subsequences as well as further configurations, each of them containing $r+1$ subsequences $1^*1 - 0^*0$. An eigenvector of H_0 with an eigenvalue α given by the diagonal element of H_0 on any configuration (47) is given by a linear combination of this configuration and all others generated from it by repeated action of H_0 with the diagonal element of H_0 taking a value different from α. Completely analogous statements hold true for the configurations

$$\begin{aligned}
&\big(\ldots, 2^*2, 2^*2, 2^*0, 0^*(2, 2^*)^n\big)\\
&\big(\ldots, 2^*2, 2^*2, 2^*(0, 2^*)^{r_R}(2, 2^*)^{n_R}(0, 2^*)^{r_{R-1}}(2, 2^*)^{n_{R-1}}\ldots(0, 2^*)^{r_1}(2, 2^*)^{n_1}(0, 0^*)^k(2, 2^*)^{n_0}\big)\\
&\big(\ldots, 2^*2, 2^*2, 2^*(0, 0^*)^k(2, 2^*)^{n_0}(2, 0^*)^{r_R}(2, 2^*)^{n_R}(2, 0^*)^{r_{R-1}}(2, 2^*)^{n_{R-1}}\ldots(2, 0^*)^{r_1}(2, 2^*)^{n_1}\big)
\end{aligned}$$

$$\big(\ldots,2^*2,2^*2,2^*(0,2^*)^{r_R}(2,2^*)^{n_R}(0,2^*)^{r_{R-1}}(2,2^*)^{n_{R-1}}\ldots$$

$$\ldots(0,2^*)^{r_1}(2,2^*)^{n_1}(0,0^*)^k(2,2^*)^{n_0}(2,0^*)^{k_S}(2,2^*)^{m_S}(2,0^*)^{k_{S-1}}(2,2^*)^{m_{S-1}}\ldots(2,0^*)^{k_1}(2,2^*)^{m_1}\big) \quad (48)$$

with $R,S>0$; $k=0,1$; $n,n_0,n_1,m_1\geq 0$; $n_l,m_l>0$ for $l>1$ and $r_l,k_l>0\;\forall l$.

So fare, (33), (35) and the second line of equations (32) do not enter the evaluation of H_0. A simple example requiring (33) in addition is given by

$$H_0\Big(\big(\ldots,2^*2,2^*2,2^*1,1^*(2,2^*)^n\big)\Big)=-(2n+1)\big(\ldots,2^*2,2^*2,2^*1,1^*(2,2^*)^n\big)$$

$$+n\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^{n-1},2^*\big)+(n+1)\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^n,2^*\big)$$

$$-n\big(\ldots,2^*2,2^*2,2^*1,(1^*1-0^*0),1^*(2,2^*)^{n-1}\big)-(n+1)\big(\ldots,2^*2,2^*2,2^*1,(1^*1-0^*0),1^*(2,2^*)^n\big)$$

$$-2\sum_{m=0}^{n-2}(m+1)\,\big(\ldots,2^*2,2^*2,2^*1,1^*2,(2^*2)^{n-m-2},(1^*1-0^*0),(2^*2)^m,2^*\big)$$

$$-2\sum_{m=0}^{\infty}(n+m+2)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^m,2^*1,1^*(2,2^*)^n\big) \quad (49)$$

The second line shows that configurations of the type (44) with $i=2$ are generated by repeated action of H_0 on $\big(\ldots,2^*2,2^*2,2^*1,1^*(2,2^*)^n\big)$. As the following example reveals, application of H_0 on configurations of the rhs of (49) may produce several configurations with one subsequence 1^*1-0^*0 in addition to those with two such subsequences:

$$H_0\Big(\big(\ldots,2^*2,2^*1,(1^*1-0^*0),1^*(2,2^*)^n\big)\Big)=-2(n+1)\big(\ldots,2^*2,2^*1,(1^*1-0^*0),1^*(2,2^*)^n\big)$$

$$+n\big(\ldots,2^*2,2^*2,2^*1,1^*2,(1^*1-0^*0),2^*(2,2^*)^{n-1}\big)+(n+2)\big(\ldots,2^*2,2^*2,(1^*1-0^*0),2^*1,1^*(2,2^*)^n\big)$$

$$-n\big(\ldots,2^*2,2^*2,2^*1,(1^*1-0^*0)^2,1^*(2,2^*)^{n-1}\big)-(n+2)\big(\ldots,2^*2,2^*2,2^*1,(1^*1-0^*0)^2,1^*(2,2^*)^n\big)$$

$$-2\sum_{m=0}^{n-2}(m+1)\,\big(\ldots,2^*2,2^*2,2^*1,(1^*1-0^*0),1^*2,(2^*2)^{n-m-2},(1^*1-0^*0),(2^*2)^m,2^*\big)$$

$$-2\sum_{m=0}^{\infty}(n+m+3)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^m,2^*1,(1^*1-0^*0),1^*(2,2^*)^n\big) \quad (50)$$

In (49) and (50), the fourth line is not present for $n=0,1$. Similar shifts of pieces 1^*1-0^*0 occur for configurations arising from the action of H_0 on $\big(\ldots,2^*2,2^*2,2^*1,i^*2,(2^*2)^n,2^*\big)$ and $\big(\ldots,2^*2,2^*2,1^*i,(2^*2)^n,2^*\big)$ with $i=0,2$, for example. They may be summarized by

$$H_0\Big(\big(\ldots i,(1^*1-0^*0)^l,j^*i_n,j_n^*i_{n-1},\ldots i_2,j_2^*i_1,j_1^*\big)\Big)=$$

$$\ldots+\delta_{i,1}(n+l+1)\big(\ldots 2,(1^*1-0^*0),2^*1,(1^*1-0^*0)^{l-1},j^*i_n,j_n^*i_{n-1},\ldots i_2,j_2^*i_1,j_1^*\big)+\ldots$$

$$H_0\Big(\big(\ldots j,(1^*1-0^*0)^l,i^*2,2^*i_n,j_n^*i_{n-1},\ldots i_2,j_2^*i_1,j_1^*\big)\Big)=$$

$$\ldots+\delta_{i,1}(n+1)\big(\ldots j,(1^*1-0^*0)^{l-1},1^*2,(1^*1-0^*0),2^*i_n,j_n^*i_{n-1},\ldots i_2,j_2^*i_1,j_1^*\big)+\ldots \quad (51)$$

The contribution on the rhs of (51) applies to all $i,i_1,j_1,i_2,j_2,\ldots,i_N,j_N=0,1,2$. Left of the subsequences indicated here, any sequence with almost all entries equal to 2 or 2^* may be inserted. The limit (33) also enters the analysis of the set of configurations $\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*i_2,j_2^*(2,2^*)^n\big)$, $n=0,1,2,\ldots$, with the cases $i_1=j_1=2$, $i_2=j_2=2$ and $j_1=i_2=0,1$ excluded:

$$H_0\Big(\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*i_2,j_2^*(2,2^*)^n\big)\Big)=$$

$$-\Big((n+2)\big(1+y_{j_1,i_1,2}\big)+(n+1)\big(y_{i_1,j_1,i_2}+y_{j_2,i_2,j_1}\big)+n\big(1+y_{i_2,j_2,2}\big)\Big)\cdot\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*i_2,j_2^*(2,2^*)^n\big)$$

$$+\delta_{i_1,1}\delta_{j_1,1}(n+2)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),2^*i_2,j_2^*(2,2^*)^n\big)$$

$$+\delta_{i_2,1}\delta_{j_2,1}\,n\,\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*2,(1^*1-0^*0),(2^*2)^{n-1},2^*\big)$$

$$-(n+2)\,\big(\ldots,2^*2,2^*2,2^*i_1,(1^*1-0^*0),j_1^*i_2,j_2^*(2,2^*)^n\big)$$

$$-n\,\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*i_2,(1^*1-0^*0),j_2^*(2,2^*)^{n-1}\big)$$

$$-2\sum_{m=0}^{n-2}(m+1)\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*i_2,j_2^*2,(2^*2)^{n-m-2},(1^*1-0^*0),(2^*2)^m,2^*\big)$$

$$-2\sum_{m=0}^{\infty}(n+m+3)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^m,2^*i_1,j_1^*i_2,j_2^*(2,2^*)^n\big) \tag{52}$$

For none of the configurations considered so fare, the (repeated) action of H_0 creates configurations with a subsequence 1^*1+0^*0. Hence, (30), (31), (33), (34) and the first of equations (32) are sufficient for its analysis. The following example involving also (35) and the remainder of (32) completes the evaluation of $\big(\ldots,2^*2,2^*2,2^*i_1,j_1^*i_2,j_2^*(2,2^*)^n\big)$:

$$H_0\Big(\big(\ldots,2^*2,2^*2,2^*i,(1^*1+0^*0),j^*(2,2^*)^n\big)\Big)=$$

$$-\big(2n+2-\delta_{i,2}+\delta_{j,2}\big)\big)\cdot\big(\ldots,2^*2,2^*2,2^*i,(1^*1+0^*0),j^*(2,2^*)^n\big)$$

$$+\delta_{i,1}(n+2)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),2^*1,j^*(2,2^*)^n\big)$$

$$+2(\delta_{i,1}+\delta_{j,1})(n+1)\big(\ldots,2^*2,2^*2,2^*i,(1^*1-0^*0),j^*(2,2^*)^n\big)$$

$$+2(1-\delta_{i,1})(n+1)\,\big(\ldots,2^*2,2^*2,2^*1,1^*i,j^*(2,2^*)^n\big)$$

$$+2(1-\delta_{j,1})(n+1)\,\big(\ldots,2^*2,2^*2,2^*i,j^*1,1^*(2,2^*)^n\big)$$

$$+2(n+1)\,\big(\ldots,2^*2,2^*2,2^*i,j^*(2,2^*)^n\big)$$

$$+2(n+1)\,\big(\ldots,2^*2,2^*2,2^*i,j^*(2,2^*)^{n+1}\big)$$

$$+\delta_{j,1}n\,\big(\ldots,2^*2,2^*2,2^*i,1^*2,(1^*1-0^*0),(2^*2)^{n-1},2^*\big)$$

$$-(n+2)\,\big(\ldots,2^*2,2^*2,2^*i,(1^*1-0^*0),(1^*1+0^*0),j^*(2,2^*)^n\big)$$

$$-n\,\big(\ldots,2^*2,2^*2,2^*i,(1^*1+0^*0),(1^*1-0^*0),1^*(2,2^*)^{n-1}\big)$$

$$-2\sum_{m=0}^{n-2}(m+1)\,\big(\ldots,2^*2,2^*2,2^*i,(1^*1+0^*0),j^*2,(2^*2)^{n-m-2},(1^*1-0^*0),(2^*2)^m,2^*\big)$$

$$-2\sum_{m=0}^{\infty}(n+m+3)\,\big(\ldots,2^*2,2^*2,(1^*1-0^*0),(2^*2)^m,2^*i,(1^*1+0^*0),j^*(2,2^*)^n\big) \tag{53}$$

In the fifth and sixth line of (53), configurations included in (52) appear. The configuration in the fourth line is found in $H_0\Big(\big(\ldots,2^*2,2^*2,2^*i,j^*(2,2^*)^n\big)\Big)$ which also appears in the seventh line of (53). As in the previous examples, the

configuration on the lhs of (53) is not generated by (repeated) action of H_0 on any of the other configurations found in the rhs of the same equation. Further investigation along these lines suggests to formulate this observation more generally. This is conveniently done by means of the notation

$$H_0\Big(\big(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*\big)\Big) = \sum_{i'_l, j'_l = 0,1,2} h_{\ldots i_3,\, j_3^* i_2,\, j_2^* i_1,\, j_1^*;\, \ldots i'_3, j'^*_3 i'_2, j'^*_2 i'_1, j'^*_1} \cdot \big(\ldots i'_3, j'^*_3 i'_2, j'^*_2 i'_1, j'^*_1\big) \tag{54}$$

Only finitely many values of i_l, i'_l, j_l, j'_l differ from 2 according to the boundary condition adopted here. The following statement applies to all configurations $\big(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*\big)$ considered above:

$$h_{\ldots i_3,\, j_3^* i_2,\, j_2^* i_1,\, j_1^*;\, \ldots i'_3, j'^*_3 i'_2, j'^*_2 i'_1, j'^*_1} \neq 0 \implies$$

$$h_{\ldots i'_3,\, j'^*_3 i'_2,\, j'^*_2 i'_1,\, j'^*_1;\, \ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*} = \beta(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*) \cdot \prod_{l=1}^{\infty} \delta_{i_l, i'_l} \delta_{j_l, j'_l} \tag{55}$$

with

$$\beta(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*) = -\sum_{l=1}^{\infty} \big(y_{i_{l+1}, j_{l+1}, i_l} + y_{j_l, i_l, j_{l+1}}\big) \tag{56}$$

and $y_{i,j,k}$ given by (31). The function $\beta(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)$ is well defined due to (31) and the restriction $i_l = j_l = 2$ for almost all l. As the analysis reveals, (55) remains valid for a large set of configurations including the collection of all $\big(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*\big)$ which do not contain any of the subsequences

$$\begin{gathered} (1^*1 + 0^*0), (1^*1 - 0^*0) \\ (1^*1 - 0^*0), (1^*1 + 0^*0) \\ 0^*1, 1^*0 \end{gathered} \tag{57}$$

Moreover, the elements of H_0 with respect to this restricted set of configurations can be arranged as a trigonal matrix. This can be seen introducing a number $\Omega\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) \in \mathbf{Z}_{0,+}$ for each configuration $\big(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*\big)$:

$$\begin{aligned} &\Omega\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) \equiv \\ &\Omega^-\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) + \tilde{\Omega}^-\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) - \Omega^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) - \tilde{\Omega}^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) \\ &\qquad -\omega^-\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) - \omega^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) \end{aligned} \tag{58}$$

Here $\Omega^-\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big)$ $\Big(\Omega^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big)\Big)$ denotes the number of subsequences $1^*1 - 0^*0$ $\Big(1^*1 + 0^*0\Big)$ found in the configuration $\big(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*\big)$ and $\frac{1}{2}\Big(\tilde{\Omega}^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) + \tilde{\Omega}^-\Big(\big(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*\big)\Big)\Big)$ and $\frac{1}{2}\Big(\tilde{\Omega}^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) - \tilde{\Omega}^-\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big)\Big)$ count the subsequences $1^*2, 2^*1$ and $0^*2, 2^*0$, respectively. To each subsequence

$$\begin{aligned} &j_1^* i_1, (1^*1 + 0^*0)^{m_1}, j_1^{*(1)}1, j_2^{*(1)}1, \ldots, j_{t_1}^{*(1)}1, (1^*1 + 0^*0)^{m_2}, j_1^{*(2)}1, j_2^{*(2)}1, \ldots, j_{t_2}^{*(2)}1, (1^*1 + 0^*0)^{m_3}, \ldots \\ &\qquad \ldots, (1^*1 + 0^*0)^{m_S}, j_1^{*(s)}1, j_2^{*(s)}1, \ldots, j_{t_S}^{*(s)}1, (1^*1 + 0^*0)^{m_{S+1}}, j_2^* i_2 \end{aligned} \tag{59}$$

with $S \geq 1$; $m_{S+1} \geq 0$; $m_l \geq 1$ for $l \leq S$; $j_{m_l}^{*(l)} = 0, 2$ for $1 \leq m_l \leq t_l$, $1 \leq l \leq S$ and the cases $i_m = j_m = 0, 1$ and $i_m = 1, j_m = 0, 2$ excluded for $m = 1, 2$, the number

$$\sum_{l=1}^{S} m_l \sum_{l'=l}^{S} t_{l'} \tag{60}$$

is assigned. Furthermore, to each subsequence

$$i_2^* j_2, (1^*1+0^*0)^{n_{R+1}}, 1^* i_{s_R}^{(R)}, \ldots, 1^* i_2^{(R)}, 1^* i_1^{(R)}, (1^*1+0^*0)^{n_R}, \ldots$$

$$\ldots, (1^*1+0^*0)^{n_3}, 1^* i_{s_2}^{(2)}, \ldots, 1^* i_2^{(2)}, 1^* i_1^{(2)}, (1^*1+0^*0)^{n_2}, 1^* i_{s_1}^{(1)}, \ldots, 1^* i_2^{(1)}, 1^* i_1^{(1)}, (1^*1+0^*0)^{n_1}, i_1^* j_1 \qquad (61)$$

with $R \geq 1$; $n_{R+1} \geq 0$; $n_l \geq 1$ for $l \leq R$; $i_{m_l}^{(l)} = 0, 2$ for $1 \leq m_l \leq s_l$, $1 \leq l \leq R$ and the above restrictions on $i_m, j_m,\ m = 1, 2$, the number

$$\sum_{l=1}^{R} n_l \sum_{l'=l}^{R} s_{l'} \qquad (62)$$

is attributed. $\omega^+\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big)$ equals the sum of numbers (60) and (62) found for the configuration $(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*))$. Similarly, the numbers

$$\sum_{l=2}^{S+1} m_l \sum_{l'=1}^{l-1} t_{l'} \qquad \text{and} \qquad \sum_{l=2}^{R+1} n_l \sum_{l'=1}^{l-1} s_{l'} \qquad (63)$$

are assigned to each subsequence obtained from (59) and (61) by replacing each term $1^*1 + 0^*0$ by $1^*1 - 0^*0$. Then $\omega^-\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big)$ denotes the sum of all numbers (63) related to such subsequences present in the configuration $(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*))$. Direct examination of the limits (32)-(35) reveals that two configurations $(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)$ and $(\ldots i'_3, j_3'^* i'_2, j_2'^* i'_1, j_1'^*)$ with $h_{\ldots i_3,\, j_3^* i_2,\, j_2^* i_1,\, j_1^*;\, \ldots i'_3, j_3'^* i'_2, j_2'^* i'_1, j_1'^*} \neq 0$ and $(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*) \neq (\ldots i'_3, j_3'^* i'_2, j_2'^* i'_1, j_1'^*)$ satisfy

$$\Omega\big((\ldots i'_3, j_3'^* i'_2, j_2'^* i'_1, j_1'^*)\big) > \Omega\big((\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)\big) \qquad (64)$$

if the configuration $(\ldots i'_3, j_3'^* i'_2, j_2'^* i'_1, j_1'^*)$ is not obtained from $(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)$ either by substituting one subsequence $\big(1^*1 + (-1)^k 0^*0\big)\big(1^*1 - (-1)^k 0^*0\big)$ by $\big(1^*1 - (-1)^k 0^*0\big)\big(1^*1 + (-1)^k 0^*0\big)$ or $0^*1, 1^*0$, where $k = 0, 1$, or by replacing one subsequence $0^*1, 1^*0$ by $(1^*1+0^*0)(1^*1-0^*0)$ or $(1^*1-0^*0)(1^*1+0^*0)$. In these cases, the limits (32), (33) and (35) yield for $i, j, i_l, j_l = 0, 1, 2$:

$$H_0\Big(\big(\ldots, i, (1^*1+0^*0), (1^*1-0^*0), j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big)\Big) =$$
$$\ldots + (n+2)\big(\ldots, i, (1^*1-0^*0), (1^*1+0^*0), j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big)$$
$$-2(n+2)\big(\ldots i, 0^*1, 1^*0, j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big) + \ldots$$
$$H_0\Big(\big(\ldots, i, (1^*1-0^*0), (1^*1+0^*0), j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big)\Big) =$$
$$\ldots + (n+1)\big(\ldots, i, (1^*1+0^*0), (1^*1-0^*0), j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big)$$
$$-2(n+1)\big(\ldots i, 0^*1, 1^*0, j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big) + \ldots \qquad (65)$$

In both cases, the first contribution stems from (32) and the second from (35). Furthermore,

$$H_0\Big(2\big(\ldots i, 0^*1, 1^*0, j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big)\Big) =$$
$$\ldots + (n+2)\big(\ldots, i, (1^*1-0^*0), (1^*1+0^*0), j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big)$$
$$+(n+1)\big(\ldots, i, (1^*1+0^*0), (1^*1-0^*0), j^* i_n, j_n^* i_{n-1}, \ldots i_2, j_2^* i_1, j_1^*\big) + \ldots \qquad (66)$$

where both contributions arise from (33). As in (51), left of the entry i any sequence with almost all entries fixed by 2 or 2^* may be chosen. Together with the limits (30), (31), the property (64) implies triangularity of the matrix formed by the elements of H_0 for the restricted set of configurations if arranged in an order indicated by increasing numbers $\Omega(\ldots i_3, j_3^* i_2, j_2^* i_1, j_1^*)$.

C. A particular set of subsequences

This subsection specializes on a set of configurations for which the matrix elements of H_0 don't show a triangular form in the basis (40). As apparent from (65), (66), H_0 couples configurations which differ only by maximal subsequences consisting of $K-L$ terms (1^*1+0^*0), $M-L$ terms (1^*1-0^*0) and L terms $=0^*1, 1^*0$ for fixed pairs $K, M \geq 1$ and $0 \leq L \leq \min(K,M)$. Explicitly, these configurations are written

$$\big(\ldots, j^*_{n+3}i_{n+2}, j^*_{n+2}i_{n+1}, (0^*1, 1^*0)^{n_1}, j^*_{n+1}i_n, j^*_n i_{n-1}, \ldots, j^*_2 i_1, j^*_1\big)$$

with $n \geq 0$ and $n_1 \geq 1$ or

$$\Big(\ldots, j^*_{n+3}i_{n+2}, j^*_{n+2}i_{n+1}, (0^*1,1^*0)^{n_0}\Big\{\prod_{t_1=1}^{s_1}(1^*1+(-1)^{\sigma_{t_1}}0^*0)\Big\}\,(0^*1,1^*0)^{n_1}\Big\{\prod_{t_2=1}^{s_2}(1^*1+(-)^{\sigma_{t_2}}0^*0)\Big\},\ldots$$

$$\ldots,(0^*1,1^*0)^{n_{R-1}}\Big\{\prod_{t_R=1}^{s_R}(1^*1+(-1)^{\sigma_{t_R}}0^*0)\Big\}\,(0^*1,1^*0)^{n_R}, j^*_{n+1}i_n, j^*_n i_{n-1},\ldots, j^*_2 i_1, j^*_1\Big) \tag{67}$$

where $R \geq 1$; $n, n_0, n_R \geq 0$; $n_l \geq 1$ for $0 < l < R$; $s_l \geq 1$ and $\sigma_{t_l} = 0, 1$. In (67), the choices $j_{n+2} = i_{n+1} = 0, 1$ or $j_{n+1} = i_n = 0, 1$ as well as $j_{n+3} = i_{n+1} = 0$, $i_{n+2} = j_{n+2} = 1$ or $j_{n+1} = i_{n-1} = 0$, $i_n = j_n = 1$ are excluded. Any diagonal element of H_0 for the configurations (67) depends on n, n_l and σ_{t_l} and on the remaining entries i_l, j_l. The contribution of the latter is the same for all diagonal elements and will be denoted by $-\kappa$ below. As an example, equations (30) and(31) yield $\kappa = 2n+1+K+M$ for the configuration $\big(\ldots, 2^*2, 2^*2, (1^*1+0^*0)^K, (1^*1-0^*0)^M, (2^*2)^n, 2^*\big)$. The collection of matrix elements of H_0 with respect to all configurations (67) for a given set of values n, κ and K, M may be called $h^{(n,\kappa)}$. For the configurations (67), the diagonal elements of $h^{(n,\kappa)}$ are given by

$$-\kappa - n_1(2n_1+2n+1) \tag{68}$$

for the first configuration in (67) and by

$$-\kappa - n_R(2n_R+2n+1) - \sum_{l=1}^{R} n_{l-1}\Big(2n+1+2n_{l-1}+2\sum_{l'=l}^{R}(2n_{l'}+s_{l'})\Big) \tag{69}$$

for all others. All other nonvanishing elements of $h^{(n,\kappa)}$ follow directly from (65) and (66). Within the subsequences (67), the symbols $+$, $-$ and $\circ$ will be used below to abbreviate the terms $1^*1+0^*0, 1^*1-0^*0$ and twice the term $0^*1, 1^*0$, respectively. Explicit reference to the entries i_l, j_l in the configurations (67) will be omitted in most of the remainder of this section. Then the dependence on n and $\{i_l, j_l\}$ will be reminded by a subscript n, κ. Equations (65), (66) imply

$$h^{(n,\kappa)}_{\left(b\left\{\circ-(-+)\right\}a\right);\ \left(b\left\{\circ+(-+)\right\}a\right)} = h^{(n,\kappa)}_{\left(b\left\{\circ-(-+)\right\}a\right);\ \left(b+-a\right)} = 0$$

$$h^{(n,\kappa)}_{\left(b\left\{\circ-(+-)\right\}a\right);\ \left(b\left\{\circ+(+-)\right\}a\right)} = h^{(n,\kappa)}_{\left(b\left\{\circ-(+-)\right\}a\right);\ \left(b-+a\right)} = 0 \tag{70}$$

where a, b abbreviate parts of a subsequence built from $+$, $-$ and $\circ$, $\big(b\{\circ \pm (+-)\}a\big)_{n,\kappa} = \big(b\circ a\big)_{n,\kappa} \pm \big(b+-a\big)_{n,\kappa}$, $\big(b\{\circ\pm(-+)\}a\big)_{n,\kappa} = \big(b\circ a\big)_{n,\kappa} \pm \big(b-+a\big)_{n,\kappa}$ and the notation $h^{(n,\kappa)}\big((c)_{n,\kappa}\big) = \sum_{c'} h^{(n,\kappa)}_{(c);\,(c')}(c')_{n,\kappa}$ with subsequences c, c' composed from $+$, $-$ and $\circ$ is used. With (70), (65), (66) and (68), (69) it is easily verified that the configurations

$$\Big((-)^{M-1}\Big\{\circ-\big(-+\big)\Big\}(+)^{K-1}\Big)_{n,\kappa}$$

and

$$\Big((-)^{r_0}\Big\{\circ-\big(-+\big)\Big\}(+)^{s_0}\Big\{\circ-\big(+-\big)\Big\}(-)^{r_1}\Big\{\circ-\big(-+\big)\Big\}(+)^{s_1}\Big\{\circ-\big(+-\big)\Big\}(-)^{r_2}\ldots$$

$$\ldots(+)^{s_{m-1}}\Big\{\circ-\big(+-\big)\Big\}(-)^{r_m}\Big\{\circ-\big(-+\big)\Big\}(+)^{s_m}\Big)_{n,\kappa} \tag{71}$$

with $m = 1, 2, 3, \ldots$; $r_l, s_l = 0, 1, 2, \ldots$ and $r_0 = M - 2m - 1 - \sum_{l=1}^{m} r_l \geq 0$; $s_0 = K - 2m - 1 - \sum_{l=1}^{m} s_l \geq 0$ are eigenvectors of $h^{(n,\kappa)}$ with eigenvalues

$$-(\kappa + n + K + 1) \qquad \text{and} \qquad -\Big(\kappa + 4m^2 + m(n+1) + (m+1)(n+2) + K - 1 + 2\sum_{l=1}^{m} l(r_l + s_l)\Big). \tag{72}$$

Similarly, the configurations

$$\Big((-)^{\tilde r_0}\Big\{\circ - (-+)\Big\}(+)^{\tilde s_0}\Big\{\circ - (+-)\Big\}(-)^{r_1}\Big\{\circ - (-+)\Big\}(+)^{s_1}\Big\{\circ - (+-)\Big\}(-)^{r_2}\ldots$$
$$\ldots(+)^{s_{m-1}}\Big\{\circ - (+-)\Big\}(-)^{r_m}\Big)_{n,\kappa} \tag{73}$$

with $M - 2m - \sum_{l=1}^{m} r_l = \tilde r_0 \geq 0$; $K - 2m - \sum_{l=1}^{m-1} s_l = \tilde s_0 \geq 0$ are eigenvectors with eigenvalues

$$-\Big(\kappa + 2(m-1)(2m-1) + m(n+1) + m(n+2) + K + 2\sum_{l=1}^{m} l r_l + 2\sum_{l=1}^{m-1} l s_l\Big) \tag{74}$$

Exchanging $+$ and $-$ in (71) and (73) as well as K and M in the restrictions for r_0, s_0 and $\tilde r_0, \tilde s_0$ yields further eigenvectors. The eigenvalues for the eigenvectors resulting from (71) follow from (72) by the replacement $K \to M-1$ while those for the eigenvectors obtained from (73) follow from (74) by the substitution $K \to M$.

Further analysis reveals that the matrix elements of $h^{(n,\kappa)}$ exhibit a triangular structure with respect to a suitable basis of configurations composed from $+$, $-$, $\big\{\circ - (-+)\big\}$, $\big\{\circ - (+-)\big\}$ and $\circ$ such that $+$ and $-$ are adjacent only as parts of the linear combinations $\big\{\circ - (-+)\big\}$ and $\big\{\circ - (+-)\big\}$. Such a basis configuration may be written

$$\Big((\circ)^{n_0}\, c_1\, (\circ)^{n_1}\, c_2\, (\circ)^{n_2}\, c_3\, (\circ)^{n_3}\ldots c_{R-1}\, (\circ)^{n_{R-1}}\, c_R\, (\circ)^{n_R}\Big)_{n,\kappa} \tag{75}$$

where $n_0, n_R \geq 0$; $n_l > 0$ for $0 < l < R$ and the parts c_l contain any sequence of $+$, $-$, $\big\{\circ - (-+)\big\}$ and $\big\{\circ - (+-)\big\}$ provided that only the terms $+$, $\big\{\circ - (-+)\big\}$ or $\big\{\circ - (+-)\big\}$ are neighbors of $+$ and only $-$, $\big\{\circ - (-+)\big\}$ or $\big\{\circ - (+-)\big\}$ are next to $-$. For an arbitrary part a of c_l, the length $\lambda(a)$ counts once each symbol $+$ or $-$ found in a and twice each linear combination $\big\{\circ - (-+)\big\}$ or $\big\{\circ - (+-)\big\}$ there. To any decomposition $c_l = \big(b\big\{\circ - \big(\rho, -\rho\big)\big\}a\big)$ with $\rho = \pm$ the number $n + 1 + \delta_{\rho,-} + \lambda(a)$ is associated. Summing these numbers for all such decompositions of c_l yields a number $\gamma(c_l)$. In terms of these notations, (65), (66) and (68), (69) yield the diagonal matrix elements referring to the basis configurations (75):

$$h^{(n,\kappa)}_{\big((\circ)^{n_0}\, c_1\, (\circ)^{n_1}\, c_2\, (\circ)^{n_2}\ldots c_R\, (\circ)^{n_R}\big);\, \big((\circ)^{n_0}\, c_1\, (\circ)^{n_1}\, c_2\, (\circ)^{n_2}\ldots c_R\, (\circ)^{n_R}\big)} = -\kappa - \sum_{l=1}^{R} \gamma(c_l) \tag{76}$$

One type of nondiagonal elements of $h^{(n,\kappa)}$ in the basis (75) amounts to exchanging one linear combination $\big\{\circ - (\rho, -\rho)\big\}$ with a neighboring a single symbol $+$ or $-$:

$$h^{(n,\kappa)}_{\big(b\{\circ-(-+)\}-a\big);\, \big(b-\{\circ-(-+)\}a\big)} = n + 2 + \lambda(a) \qquad b \neq (\ldots +$$
$$h^{(n,\kappa)}_{\big(b+\{\circ-(-+)\}a\big);\, \big(b\{\circ-(-+)\}+a\big)} = n + 3 + \lambda(a) \qquad a \neq -\ldots)$$
$$h^{(n,\kappa)}_{\big(b\{\circ-(+-)\}+a\big);\, \big(b+\{\circ-(+-)\}a\big)} = n + 1 + \lambda(a) \qquad b \neq (\ldots -$$
$$h^{(n,\kappa)}_{\big(b-\{\circ-(+-)\}a\big);\, \big(b\{\circ-(+-)\}-a\big)} = n + 2 + \lambda(a) \qquad a \neq +\ldots) \tag{77}$$

In the cases excluded in the right column of (77), the following matrix elements are found:

$$h^{(n,\kappa)}_{\big(b+\{\circ-(-+)\}-a\big);\, \big(b\circ\{\circ-(-+)\}a\big)} = -h^{(n,\kappa)}_{\big(b+\{\circ-(-+)\}-a\big);\, \big(b\{\circ-(+-)\}\{\circ-(-+)\}a\big)} = n + 2 + \lambda(a)$$

$$h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}-a\right);\ \left(b\left\{\circ-(-+)\right\}\circ a\right)} = -h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}-a\right);\ \left(b\left\{\circ-(-+)\right\}\left\{\circ-(+-)\right\}a\right)} = n+4+\lambda(a)$$

$$h^{(n,\kappa)}_{\left(b-\left\{\circ-(+-)\right\}+a\right);\ \left(b\circ\left\{\circ-(+-)\right\}a\right)} = -h^{(n,\kappa)}_{\left(b-\left\{\circ-(+-)\right\}+a\right);\ \left(b\left\{\circ-(-+)\right\}\left\{\circ-(+-)\right\}a\right)} = n+1+\lambda(a)$$

$$h^{(n,\kappa)}_{\left(b-\left\{\circ-(+-)\right\}+a\right);\ \left(b\left\{\circ-(+-)\right\}\circ a\right)} = -h^{(n,\kappa)}_{\left(b-\left\{\circ-(+-)\right\}+a\right);\ \left(b\left\{\circ-(+-)\right\}\left\{\circ-(-+)\right\}a\right)} = n+3+\lambda(a) \tag{78}$$

A second type of nondiagonal matrix elements reduces higher powers of $\left\{\circ-(\rho,-\rho)\right\}$ according to

$$h^{(n,\kappa)}_{\left(b\left\{\circ-(-+)\right\}^2 a\right);\ \left(b-\left\{\circ-(-+)\right\}+a\right)} = -\big(n+3+\lambda(a)\big) \qquad b\neq(\ldots+,\quad a\neq-\ldots)$$

$$h^{(n,\kappa)}_{\left(b\left\{\circ-(+-)\right\}^2 a\right);\ \left(b+\left\{\circ-(+-)\right\}-a\right)} = -\big(n+2+\lambda(a)\big) \qquad b\neq(\ldots-,\quad a\neq+\ldots) \tag{79}$$

The corresponding matrix elements in the cases $a=-\ldots)$ or $b=(\ldots+$ are

$$h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}^2 a\right);\ \left(b\circ\left\{\circ-(-+)\right\}+a\right)} = -h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}^2 a\right);\ \left(b\left\{\circ-(+-)\right\}\left\{\circ-(-+)\right\}+a\right)} = -\big(n+3+\lambda(a)\big)$$

$$h^{(n,\kappa)}_{\left(b\left\{\circ-(-+)\right\}^2 -a\right);\ \left(b-\left\{\circ-(-+)\right\}\circ a\right)} = -h^{(n,\kappa)}_{\left(b\left\{\circ-(-+)\right\}^2 -a\right);\ \left(b-\left\{\circ-(-+)\right\}\left\{\circ-(+-)\right\}a\right)} = -\big(n+4+\lambda(a)\big) \tag{80}$$

where $a\neq-\ldots)$ in the first and $b\neq(\ldots+$ in the second line, and

$$h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}^2 -a\right);\ \left(b\circ\left\{\circ-(-+)\right\}\circ a\right)} = -h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}^2 -a\right);\ \left(b\circ\left\{\circ-(-+)\right\}\left\{\circ-(+-)\right\}a\right)} =$$

$$= -h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}^2 -a\right);\ \left(b\left\{\circ-(+-)\right\}\left\{\circ-(-+)\right\}\circ a\right)} = h^{(n,\kappa)}_{\left(b+\left\{\circ-(-+)\right\}^2 -a\right);\ \left(b\left\{\circ-(+-)\right\}\left\{\circ-(-+)\right\}\left\{\circ-(+-)\right\}a\right)} =$$

$$= -\big(n+4+\lambda(a)\big) \tag{81}$$

Exchanging $+$ and $-$ in (80) and (81) and replacing $\lambda(a)$ by $\lambda(a)-1$ in the rightmost parts of the equations covers the cases excluded in the second line of (79). Finally, a third type of nondiagonal matrix elements substitutes a single symbol $\circ$ by the linear combinations $\left\{\circ-(-+)\right\}$ and $\left\{\circ-(+-)\right\}$:

$$h^{(n,\kappa)}_{\left(b\circ a\right);\ \left(b\left\{\circ-(-+)\right\}a\right)} = -\big(n+2+\lambda(a)\big)$$

$$h^{(n,\kappa)}_{\left(b\circ a\right);\ \left(b\left\{\circ-(+-)\right\}a\right)} = -\big(n+1+\lambda(a)\big) \tag{82}$$

Equations (77)-(82) characterize the action of $h^{(n,\kappa)}$ on any subsequence of the form (75). A particular set of subsequences built from $+$, $-$, $\left\{\circ-(-+)\right\}$, $\left\{\circ-(+-)\right\}$ and $\circ$ is generated by the action of $h^{(n,\kappa)}$ on

$$\Big((-)^{M-1}\Big\{\circ-\big(+-\big)\Big\}(+)^{K-1}\Big)_{n,\kappa}$$

and

$$\Big((-)^{r_0}\Big\{\circ-\big(+-\big)\Big\}(+)^{s_0}\Big\{\circ-\big(-+\big)\Big\}(-)^{r_1}\Big\{\circ-\big(+-\big)\Big\}(+)^{s_1}\Big\{\circ-\big(-+\big)\Big\}(-)^{r_2}\ldots$$

$$\ldots(+)^{s_{m-1}}\Big\{\circ-\big(-+\big)\Big\}(-)^{r_m}\Big\{\circ-\big(+-\big)\Big\}(+)^{s_m}\Big)_{n,\kappa} \tag{83}$$

with $m,r_m,r_l,s_l>0$ for $0<l<m;\ s_m\geq 0$, $r_0=M-2m-1-\sum_{l=1}^m r_l\geq 0$, $s_0=K-2m-1-\sum_{l=1}^m s_l>0$ and

$$\Big((-)^{\tilde r_0}\Big\{\circ-\big(+-\big)\Big\}(+)^{\tilde s_0}\Big\{\circ-\big(-+\big)\Big\}(-)^{r_1}\Big\{\circ-\big(+-\big)\Big\}(+)^{s_1}\Big\{\circ-\big(-+\big)\Big\}(-)^{r_2}\ldots$$

$$\dots(+)^{s_{m-1}}\Big\{\circ-(-+)\Big\}(-)^{r_m}\Big)_{n,\kappa} \tag{84}$$

with $m, r_l, s_l > 0$ for $0 < l < m$; $r_m \geq 0$, $\tilde{r}_0 = M - 2m - \sum_{l=1}^{m} r_l \geq 0$ and $\tilde{s}_0 = K - 2m - \sum_{l=1}^{m-1} s_l > 0$ as well as on the subsequences obtained from (83), (84) via the replacements $+ \leftrightarrow -$ and $K \leftrightarrow M$. A subsequence of the type (83) or (84) does not arise in the action of $h^{(n,\kappa)}$ on any other subsequence. The subsequences created by repeated action of $h^{(n,\kappa)}$ on the subsequences (83), (84) all have the form (75) and may be classified as follows. A first group consists of all subsequences (75) with $R = 1$ and $n_0 = n_1 = 0$. These are subsequences without a single symbol $\circ$. The second group collects all configurations with one single $\circ$ contained in one of the parts

$$\big(-\rho_1\big)^{u-u_1}\Big\{\circ-\big(\rho_1,-\rho_1\big)\Big\}\big(-\rho_1\big)^{u_1}\circ\big(-\rho_2\big)^{u_1'}\Big\{\circ-\big(-\rho_2,\rho_2\big)\Big\}\big(-\rho_2\big)^{u'-u_1'}$$

$$\Big\{\circ-\big(\rho_2,-\rho_2\big)\Big\}\big(\rho_2\big)^{r}\circ\big(-\rho_2\big)^{u_1'}\Big\{\circ-\big(-\rho_2,\rho_2\big)\Big\}\big(-\rho_2\big)^{u'-u_1'}$$

$$\big(-\rho_1\big)^{u-u_1}\Big\{\circ-\big(\rho_1,-\rho_1\big)\Big\}\big(-\rho_1\big)^{u_1}\circ\big(\rho_1\big)^{r}\Big\{\circ-\big(-\rho_1,\rho_1\big)\Big\} \tag{85}$$

Here $\rho_1, \rho_2 = \pm$, $r \geq 1$; $u, u' \geq 0$ and $0 \leq u_1 \leq u$, $0 \leq u_1' \leq u'$. Remaining parts of subsequences larger than (85) are filled by c_1 and/or c_2 with the choices $\dots\rho_1$ and $\rho_2\dots$ excluded for the left and right part in the case $u_1 \neq u$ or $u_1' \neq u'$, respectively. The third group lists all subsequences with two single symbols $\circ$. They may be contained in two parts chosen arbitrarily among (85) or in one part $\circ(\rho_3)^{k_1}\big\{\circ-(\rho,-\rho)\big\}(\rho_4)^{k_2}\circ$. The latter are obtained by superimposing two parts (85) characterized by ρ_i, u, u_1, u', u_1' and $\tilde{\rho}_j, \tilde{u}, \tilde{u}_1, \tilde{u}', \tilde{u}_1'$ such that the right factor $\big\{\circ-(\rho_i,-\rho_i)\big\}$ of the left part coincides with the left factor $\big\{\circ-(-\tilde{\rho}_j,\tilde{\rho}_j)\big\}$ of the right part. Here the third expression in (85) or the first two expressions with $u_1' = u'$ may be taken for the left part and the second expression or the remaining two choices with $\tilde{u}_1 = \tilde{u}$ for the right part. For example, the first option in (85) for both parts leads to

$$\big(-\rho_1\big)^{u-u_1}\Big\{\circ-\big(\rho_1,-\rho_1\big)\Big\}\big(-\rho_1\big)^{u_1}\circ\big(-\rho_2\big)^{u'}\Big\{\circ-\big(-\rho_2,\rho_2\big)\Big\}\big(\rho_2\big)^{\tilde{u}}\circ\big(-\tilde{\rho}_2\big)^{\tilde{u}_1'}\Big\{\circ-\big(-\tilde{\rho}_2,\tilde{\rho}_2\big)\Big\}\big(-\tilde{\rho}_2\big)^{\tilde{u}'-\tilde{u}_1'} \tag{86}$$

with $u, u', \tilde{u}, \tilde{u}' \geq 0$, $0 \leq u_1 \leq u$ and $0 \leq \tilde{u}_1' \leq \tilde{u}'$. In any case, all parts c_i occupying remaining space in the subsequence are subject to the requirement specified below (85). Alternatively, the two single $\circ$ may be found in a part

$$\big(-\rho\big)^{u-u_1}\Big\{\circ-\big(\rho,-\rho\big)\Big\}\big(-\rho\big)^{u_1}\circ\big(\rho\big)^{v}\circ\big(-\rho\big)^{u_1'}\Big\{\circ-\big(-\rho,\rho\big)\Big\}\big(-\rho\big)^{u'-u_1'} \tag{87}$$

where $\rho = \pm$ and $v = 0, 1, 2, \dots$. If $u_1 \neq u$ (or $u_1' \neq u'$), any part c_1, c_2 left (right) of (87) must be different from $\dots\rho$ (or $\rho\dots$). Generally, the n-th group includes all subsequences with n single symbols $\circ$. The are distributed over the subsequence in parts (85), (87) or in parts obtained by superimposing two or more of these according to the above description. Here the part (87) may be used with $u_1' = u'$ left of another part (85), (87) and with $u_1 = u$ right of another such part. All parts c_i completing the subsequence must be chosen within the requirements stated below (85) and (87). For $K = 3, M = 4$, $K = 4, M = 3$ or $M, K > 3$, the subsequences collected this way are not linearly independent. In particular, for suitably chosen a, b, each term in the equation

$$\big(b\circ-a\big)-\big(b-\circ a\big)=\Big(b\Big\{\circ-(-+)\Big\}-a\Big)-\Big(b-\Big\{\circ-(+-)\Big\}a\Big) \tag{88}$$

is found among the subsequences specified below equation (84). Such pairs of a, b have the form

$$b=\dots\Big\{\circ-(-+)\Big\}\qquad a=\Big\{\circ-(+-)\Big\}\dots \tag{89}$$

or

$$b=\dots\Big\{\circ-(+-)\Big\}(-)^{s}\qquad a=(-)^{t}\Big\{\circ-(-+)\Big\}\dots \tag{90}$$

where $s, t = 0, 1, 2, \dots$. Similarly, each term in

$$\big(b+\circ-a\big)=\Big(b+\Big\{\circ-(-+)\Big\}-a\Big)$$

$$-\Big(b\Big\{\circ-(+-)\Big\}^2 a\Big)+\Big(b\circ\Big\{\circ-(+-)\Big\}a\Big)+\Big(b\Big\{\circ-(+-)\Big\}\circ a\Big)-\big(b\circ\circ\, a\big) \tag{91}$$

is contained in the collection described above for appropriate pairs (a, b) with the form

$$b=\ldots\Big\{\circ-(+-)\Big\},\ a=(-)^s\Big\{\circ-(-+)\Big\}\ldots \qquad \text{or} \qquad b=\ldots\Big\{\circ-(-+)\Big\}(+)^t,\ a=\Big\{\circ-(+-)\Big\}\ldots \tag{92}$$

with $s, t = 0, 1, 2, \ldots$. Finally, all terms of

$$\begin{aligned}\big(b\circ\circ-a\big)-\big(b-\circ\circ\ a\big)&=\Big(b\Big\{\circ-(-+)\Big\}^2-a\Big)-\Big(b-\Big\{\circ-(+-)\Big\}^2 a\Big)\\ &-\Big(b\circ\Big\{\circ-(-+)\Big\}-a\Big)+\Big(b-\circ\Big\{\circ-(+-)\Big\}a\Big)-\Big(b\Big\{\circ-(-+)\Big\}\circ-a\Big)+\Big(b-\Big\{\circ-(+-)\Big\}\circ a\Big)\end{aligned} \tag{93}$$

belong to the collection for suitable a and b satisfying (90). The lhs of equations (88), (91) and (93) can be removed from the groups of subsequences introduced above. The same applies to all subsequences obtained from (88)-(93) by exchanging $+ \leftrightarrow -$.

With respect to the resulting collection $\mathcal{K}'(K, M)$ of subsequences, the elements $h^{(n,\kappa)}_{(a);\,(b)}$ form a triangular matrix. This is demonstrated by means of a number $\Upsilon(a) \geq 0$ introduced for such a subsequence a. The first contribution $\Upsilon_1(a)$ to this number counts all single symbols $\circ$ of a. Writing a in the form (75), this means $\Upsilon_1(a) = \sum_{l=0}^{R} n_l$. To define $\Upsilon(a)$, each single $\circ$ in a is substituted either by $\big\{\circ-(-+)\big\}$ or $\big\{\circ-(+-)\big\}$. In each of these $2^{\Upsilon_1(a)}$ subsequences a', three types of replacement are carried out. The first type substitutes a part $-\rho\big\{\circ-(\rho,-\rho)\big\}^L\rho$ found in a' with $\rho = \pm,\ L = 1, 2, 3, \ldots$ by $\big\{\circ-(\rho,-\rho)\big\}^{L+1}$. To such a replacement, the number $4L$ is attributed. The second and third type substitutes $\rho\big\{\circ-(-\rho,\rho)\big\}^L\big\{\circ-(\rho,-\rho)\big\}$ and $\big\{\circ-(-\rho,\rho)\big\}\big\{\circ-(\rho,-\rho)\big\}^L\rho$ in a' by $\big\{\circ-(-\rho,\rho)\big\}^L\rho\big\{\circ-(\rho,-\rho)\big\}$ and $\big\{\circ-(-\rho,\rho)\big\}\rho\big\{\circ-(\rho,-\rho)\big\}^L$, respectively. The number $2L$ is associated with each of these steps. These three replacements are repeated until no part $\rho\big\{\circ-(-\rho,\rho)\big\}$ or $\big\{\circ-(\rho,-\rho)\big\}\rho$ is left. In general, starting from a given a' several subsequences $\tilde{a}$ with this property can be reached by different combinations of these steps. For any combination, $\Upsilon_2^{a',\tilde{a}}(a)$ is introduced as the sum of the numbers attributed to each replacement entailed in it. Furthermore, to any part $\ldots\big\{\circ-(\rho,-\rho)\big\}(-\rho)^r\big\{\circ-(-\rho,\rho)\big\}^{L'}(\rho)^s\big\{\circ-(\rho,-\rho)\big\}\ldots$ contained in $\tilde{a}$ with $\rho = \pm,\ r, s = 0, 1, 2, \ldots$ and $L' = 1, 2, 3, \ldots$, the number $(L'-1)^2$ is attributed. Finally, denoting by $\Upsilon_3^{a',\tilde{a}}(a)$ the sum of all these numbers, $\Upsilon(a)$ is defined by

$$\Upsilon(a) \equiv \Upsilon_1(a) + \max_{a',\tilde{a}}\Big(\Upsilon_2^{a',\tilde{a}}(a) + \Upsilon_3^{a',\tilde{a}}(a)\Big) \tag{94}$$

In particular, this number takes the value zero for all subsequences of the form (71)-(73). Equations (77)-(82) imply that $\Upsilon(b) < \Upsilon(a)$ for any two subsequences a, b of the collection $\mathcal{K}'(K, M)$ with $h^{(n,\kappa)}_{(a);\,(b)} \neq 0$ and $a \neq b$. With respect to $h^{(n,\kappa)}$, each number $h^{(n,\kappa)}_{(a);\,(a)}$ with $a \in \mathcal{K}'(K, M)$ is an eigenvalue with the corresponding eigenvector given by a linear combination of a and some $b \in \mathcal{K}'(K, M)$ with $h^{(n,\kappa)}_{(b);\,(b)} \neq h^{(n,\kappa)}_{(a);\,(a)}$ and $h^{(n,\kappa)}_{(a);\,(b)} \neq 0$ or $h^{(n,\kappa)}_{(a);\,(c_1)} h^{(n,\kappa)}_{(c_1);\,(c_2)} \ldots h^{(n,\kappa)}_{(c_m);\,(b)} \neq 0$ for some $m > 0$ and $c_m \in \mathcal{K}'(K, M)$. The coefficient of some subsequence b satisfying these properties may vanish as well. An example is provided by the coefficient of $b = \big(\big\{\circ-(-+)\big\}^2\big)$ in the linear combination starting with $a = \big(+\big\{\circ-(-+)\big\}-\big)$. The eigenvalue $h^{(n,\kappa)}_{(a);\,(a)}$ depends on n through the contribution $-\kappa$. According to (76), the remaining dependence on n is given by $-\vartheta(a)\cdot n$, where $\vartheta(a)$ denotes the number of terms $\big\{\circ-(-+)\big\}$ and $\big\{\circ-(+-)\big\}$ found in a and is restricted by $1 \leq \vartheta(a) \leq \min(K, M)$. For given M and K, the eigenvalues with the minimal or maximal value of ϑ are readily classified. All eigenvalues with $\vartheta(a) = 1$ are given by $\kappa+n+r$ with $1 \leq r \leq \max(K, M)$ and by $\kappa + n + \max(K, M) + s$ with $1 \leq s \leq \min(K, M)$. Each eigenvalue $\kappa + n + r$ is found $2r - 1$ times for $1 \leq r \leq \min(K, M)$. If $K \neq M$, the number of eigenvalues $\kappa + n + r$ with $\min(K, M) < r \leq \max(K, M)$ is given by $2\min(K, M)$. The eigenvalue $\kappa + n + \max(K; M) + s$ occurs $2\min(K, M) - 2s + 1$ times. If $M = K$, the eigenvalues with $\vartheta(a) = M$ are $\kappa + M(n + M) + r$, where $0 \leq r \leq M$. These eigenvalues are $\binom{M}{r}$-fold degenerated. A similar pattern applies to the case $\vartheta(a) = K < M$. For any set of numbers $\big\{r_l = 0, 1, 2, \ldots\big\}_{0 \leq l \leq M}$ satisfying $\sum_{l=0}^{M} r_l = M-K$ there is a set of eigenvalues $\kappa + M(n + M) + r + \sum_{l=1}^{M} l\, r_l$ with $0 \leq r \leq M$. Each of these eigenvalues is $\binom{M}{r}$-fold degenerated. Exchanging K with M yields the corresponding results in the case $K > M$. The collection $\mathcal{K}'(K, M)$

can be made a basis for all subsequences of the form (67) by adding the subsequence $\big((-)^{M-K}(\circ)^K\big)$ for $M \geq K$ and $\big((+)^{K-M}(\circ)^M\big)$ for $K \geq M$. According to (76)-(82), the nonvanishing matrix elements involving $\big((-)^{M-K}(\circ)^K\big)$ for $M \geq K$ read

$$h^{(n,\kappa)}_{\big((-)^{M-K}(\circ)^K\big);\,\big((-)^{M-K}(\circ)^K\big)} = -\kappa \qquad M \geq K$$

$$h^{(n,\kappa)}_{\big((+)^{K-M}(\circ)^M\big);\,\big((+)^{K-M}(\circ)^M\big)} = -\kappa \qquad K \geq M \tag{95}$$

and

$$h^{(n,\kappa)}_{\big((-)^{M-K}(\circ)^K\big);\,\big((-)^{M-K}(\circ)^{K-1-L}\{\circ-(\rho,-\rho)\}(\circ)^L\big)} = -(n+1+\delta_{\rho,-}+2L) \qquad M \geq K$$

$$h^{(n,\kappa)}_{\big((+)^{K-M}(\circ)^M\big);\,\big((+)^{K-M}(\circ)^{M-1-L}\{\circ-(\rho,-\rho)\}(\circ)^L\big)} = -(n+1+\delta_{\rho,-}+2L) \qquad K \geq M \tag{96}$$

for $\rho = \pm$ and $0 \leq L \leq \min(K,M)-1$. As shown in appendix A, all subsequences $\big((-)^{M-K}(\circ)^{K-1-L}\{\circ - (\rho,-\rho)\}(\circ)^L\big)$ and $\big((+)^{K-M}(\circ)^{M-1-L}\{\circ-(\rho,-\rho)\}(\circ)^L\big)$ can be written as linear combinations of subsequences contained in the collection $\mathcal{K}'(K,M)$. Obviously, $h^{(n,\kappa)}$ is trigonal with respect to the enlarged collection $\mathcal{K}(K,M) \equiv \big\{\mathcal{K}'(K,M), \big((-)^{M-K}(\circ)^K\big)\big\}$ for $M \geq K$ or $\mathcal{K}(K,M) \equiv \big\{\mathcal{K}'(K,M), \big((+)^{K-M}(\circ)^M\big)\big\}$ for $K \geq M$. An eigenvector of H_0 with the eigenvalue $h^{(n,\kappa)}_{(a);\,(a)}$ is given by a linear combination of the particular configuration containing the subsequence a and the infinitely many configurations obtained from it by repeated action of H_0 with the diagonal element of H_0 different from $h^{(n,\kappa)}_{(a);\,(a)}$.

D. General configurations

Now the structure of a half-infinite configuration $\big(\ldots, i_3j_3^*, i_2j_2^*, i_1j_1^*\big)$ with several sets of subsequences composed of $1^*1 \pm 0^*0$ and/or $0^*1, 1^*0$ can be specified. A general configuration may be written as linear combination of configurations

$$\Big(\big\{a_l(K_l,M_l)\big\}_{1\leq l\leq T},\ \big\{j_{s+1},i_s\big\}_{s>m_T},\ \big\{j_{s+1},i_s\big\}_{m_l<s\leq m_{l+1},\,1\leq l<T},\ \big\{j_{s+1},i_s\big\}_{1\leq s\leq m_1},\ j_1\Big) \equiv$$

$$\Big(\ldots i_{m_T+2}, j^*_{m_T+2}i_{m_T+1}, a_T(K_T,M_T), j^*_{m_T+1}i_{m_T}, j^*_{m_T}\ldots i_{m_3+2}, j^*_{m_3+2}i_{m_3+1}, a_3(K_3,M_3), j^*_{m_3+1}i_{m_3}, j^*_{m_3}\ldots$$

$$\ldots i_{m_2+2}, j^*_{m_2+2}i_{m_2+1}, a_2(K_2,M_2), j^*_{m_2+1}i_{m_2}, j^*_{m_2}\ldots i_{m_1+2}, j^*_{m_1+2}i_{m_1+1}, a_1(K_1,M_1), j^*_{m_1+1}i_{m_1}, j^*_{m_1}\ldots, j_2^*i_1, j_1^*\Big) \tag{97}$$

where $M_l, K_l \geq 1$, $m_{l+1} > m_l$ and $a_l(K_L,M_L)$ denotes any subsequence chosen from the basis $\mathcal{K}(K_l,M_l)$. The parts $\ldots i_{m_T+2}, j^*_{m_T+2}i_{m_T+1}$ and $j^*_{m_{l+1}+1}i_{m_{l+1}}, j^*_{m_{l+1}}\ldots i_{m_l+2}, j^*_{m_l+2}, i_{m_l+1}$ with $1 \leq l \leq T-1$ as well as $j^*_{m_1+1}i_{m_1}, j^*_{m_1}\ldots i_2, j_2^*i_1, j_1^*$ do not contain any of the terms (57). Only finitely many entries differ from 2. In analogy to the case $T=1$, a number

$$\Upsilon\Big(\big\{a_l(K_l,M_l)\big\}_{1\leq l\leq T},\ \big\{j_{s+1},i_s\big\}_{s>m_T},\ \big\{j_{s+1},i_s\big\}_{m_l<s\leq m_{l+1},\,1\leq l<T},\ \big\{j_{s+1},i_s\big\}_{1\leq s\leq m_1},\ j_1\Big) \equiv \sum_{l=1}^{T}\Upsilon\big(a_l(K_l,M_l)\big) \tag{98}$$

with $\Upsilon\big(a_l(K_l,M_l)\big)$ defined by (94) may be introduced for the configuration (97).

Two configurations $A = \Big(\big\{a'_l(K'_l,M'_l)\big\}_{1\leq l\leq T'},\ \big\{j'_{s+1},i'_s\big\}_{s>m'_T},\ \big\{j'_{s+1},i'_s\big\}_{m'_l<s\leq m'_{l+1},\,1\leq l<T'},\ \big\{j'_{s+1},i'_s\big\}_{1\leq s\leq m'_1},\ j'_1\Big)$ and $A' = \Big(\big\{a_l(K_l,M_l)\big\}_{1\leq l\leq T},\ \big\{j_{s+1},i_s\big\}_{s>m_T},\ \big\{j_{s+1},i_s\big\}_{m_l<s\leq m_{l+1},\,1\leq l<T},\ \big\{j_{s+1},i_s\big\}_{1\leq s\leq m_1},\ j_1\Big)$ related by $h_{A;\,A'} \neq 0$ can differ in five ways. First, the second configuration may be obtained from the first by replacing one subsequence $a_l(K_l,M_l) \in \mathcal{K}(K_l,M_l)$ by another subsequence $a'_l(K_l,M_l) \in \mathcal{K}(K_l,M_l)$. Then

$$\Upsilon(A') < \Upsilon(A) \tag{99}$$

Second, the two sets of subsequences $\{a_l(K_l, M_l)\}_{l\leq T}$ and $\{a'_l(K'_l, M'_l)\}_{l\leq T'}$ coincide but some entries i_s, j_s differ. Third, an additional subsequence $a(K, M) \in \mathcal{K}(K, M)$ together with a suitable change in the entries i_s, j_s may occur in the second configuration while all subsequences $a_l(K_l, M_l)$ of the first configuration are kept unchanged. Alternatively, one subsequence $a_l(K_l, M_l)$ of the first configuration is substituted by additional entries i_s, j_s in the second configuration while all remaining subsequences of the first configuration are left unchanged. The additional entries i_s, j_s don't give rise to any terms (57). All other cases involve a substitution of one subsequence $a_l(K_l, M_l) \in \mathcal{K}(K_l, M_l)$ (or of two neighboring subsequences $a_{l+1}(K_{l+1}, M_{l+1}) \in \mathcal{K}(K_{l+1}, M_{l+1})$ and $a_l(K_l, M_l) \in \mathcal{K}(K_l, M_l)$) by a subsequence from a different collection $\mathcal{K}(K'_l, M'_l)$ (or by subsequences from different collections $\mathcal{K}(K'_{l+1}, M'_{l+1})$ and $\mathcal{K}(K'_l, M'_l)$). This may be accompanied by adjustments in some entries i_s, j_s not producing any terms (57). In all cases except the first one,

$$\Omega(A') > \Omega(A) \tag{100}$$

with $\Omega(B)$ defined by (58). Hence, the elements $h_{A;\,A'}$ with A, A' of type (97), (67) or without any sequence (57) form a triangular matrix if arranged in an order indicated by (99) and (100).

To each decomposition $a_l(K_L, M_L) = \big(b'\{\circ\!-\!(\rho, -\rho)\}b\big) \in \mathcal{K}(K, M)$ the number $m_l+1+\delta_{\rho,-}+\lambda(b)+\sum_{l'=1}^{l-1}(K_{l'}+M_{l'})$ with $\rho = \pm$ and $\lambda(b)$ as defined below (75) is attributed. The sum of these numbers for all such decompositions of $a_l(K_L, M_L)$ may be denoted by $\gamma(a_l)$. Then the diagonal element of H_0 with respect to the configuration (97) is given by

$$\begin{aligned}
&-\sum_{l=1}^{T}\gamma(a_l) - \sum_{l=2}^{T}\sum_{l'=1}^{l-1}(K_{l'} + M_{l'})\left(\delta_{j_{m_l+1},2} + \delta_{i_{m_l+1},2} + \delta_{i_{m_l},0} + \delta_{j_{m_l+2},0}\right)\\
&-\sum_{l=1}^{T}\Big((m_l+1)\delta_{j_{m_l+1},2} + \big(m_l + K_l + M_l\big)\big)\,\delta_{i_{m_l+1},2} + m_l\,\delta_{i_{m_l},0} + \big(m_l + 1 + K_l + M_l\big)\big)\,\delta_{j_{m_l+2},0}\Big)\\
&-\sum_{t=1}^{m_1} t\big((1-\delta_{t,m_1})y_{i_{t+1},j_{t+1},i_t} + y_{j_t,i_t,j_{t+1}}\big)\\
&-\sum_{l=1}^{T-1}\sum_{t=m_l+1}^{m_{l+1}}\Big(t + \sum_{k=1}^{l}(K_k + M_k)\Big)\big((1-\delta_{t,m_{l+1}})y_{i_{t+1},j_{t+1},i_t} + (1-\delta_{t,m_l+1})y_{j_t,i_t,j_{t+1}}\big)\\
&-\sum_{t=m_T+1}^{\infty}\Big(t + \sum_{k=1}^{T}(K_k + M_k)\Big)\big(y_{i_{t+1},j_{t+1},i_t} + (1-\delta_{t,m_T+1})y_{j_t,i_t,j_{t+1}}\big)
\end{aligned} \tag{101}$$

If $T = 1$, the second terms of the first and third line in (101) are dropped. A configuration without terms (57) has its diagonal element given by (55) and (56). Each diagonal element of H_0 on a configuration A written in the form (97) is an eigenvalue of H_0. The corresponding eigenvector is a linear combination of A and the configurations with a different diagonal element arising from repeated action of H_0 on A.

The values of the diagonal elements (56) and (101) have upper bounds 0 and -2, respectively. Three vanishing diagonal elements are found. As stated above, the corresponding configurations are $(\ldots, 2^*2, 2^*2, i)$ with $i = 0, 1, 2$. For a fixed value of $h_{A;\,A}$, only finitely many configurations A exist. Their structure will be investigated in the following section.

VI. THE MODULE $V(\Lambda_2)$

Choosing a value N for a given configuration $(\ldots, j_3^*i_2, j_2^*i_1, j_1^*)$ such that $j_n = 2\forall n > N+1$ and $i_n = 2\forall n > N$, the numbers

$$\bar{h}_1\big((\ldots, j_3^*i_2, j_2^*i_1, j_1^*)\big) = -\delta_{j_{N+1},0} - \delta_{j_{N+1},1} + \sum_{n=1}^{N}\big(\delta_{i_n,0} + \delta_{i_n,1} - \delta_{j_n,0} - \delta_{j_n,1}\big)$$

$$\bar{h}_2\big((\dots,j_3^*i_2,j_2^*i_1,j_1^*)\big)=\delta_{j_{N+1},1}+\delta_{j_{N+1},2}+\sum_{n=1}^{N}\big(\delta_{j_n,1}+\delta_{j_n,2}-\delta_{i_n,1}-\delta_{i_n,2}\big) \tag{102}$$

may be used to define an action of h_1 and h_2 on $(\dots,j_3^*i_2,j_2^*i_1,j_1^*)$. In the following, for each configuration the numbers $\bar{h}_1$, $\bar{h}_2$ and value $\bar{H}_0$ of the diagonal element of H_0 on $(\dots,j_3^*i_2,j_2^*i_1,j_1^*)$ will be collected writing $(\bar{H}_0,\bar{h}_1,\bar{h}_2)$. For the configurations $(\dots,2^*2,2^*2,i^*)$ the definitions (102) yield $(0,0,1)$, $(0,-1,1)$ and $(0,-1,0)$. According to the remarks at the end of the last section, all other configurations have lower values $\bar{H}_0$. From (56), the configurations with the value $\bar{H}_0=-1$ are $\big(\dots,2^*2,2^*2,2^*i,j^*\big)$, $(\dots,2^*2,2^*2,1^*i,j^*)$, $(\dots,2^*2,2^*2,1^*2,j^*)$ and $(\dots,2^*2,2^*2,0^*0,j^*)$ with $i,j=0,1$. To these, (102) assigns the values $(-1,-2,0)$, $(-1,-2,1)$ and $(-1,0,k)$, $(-1,-1,k)$ with $k=0,1,2$, where $(-1,0,1)$ and $(-1,-1,0)$ are twofold and $(-1,-1,1)$ is threefold degenerated.

These values may be compared to the weight components associated to weight components of the irreducible module $V(\Lambda_2)$ at level one. The latter is characterized by a unique highest weight state $|\kappa\rangle$ with the properties

$$e_i\,|\kappa\rangle=0 \qquad\qquad h_i\,|\kappa\rangle=\delta_{i,2}\,|\kappa\rangle \qquad \text{for } i=0,1,2 \tag{103}$$

All states in the module are generated by the action of f_i on $|\kappa\rangle$. The eigenvalues of h_i on such a state may be denoted by λ_i, $i=0,1,2$. They provide the coefficients of Λ_i in the expansion of an affine weight in terms of the fundamental weights and δ. At level one, the coefficient λ_0 of Λ_0 is given by $1-\lambda_1-\lambda_2$. To specify a weight, the three coefficients are written in the form $[\lambda_0,\lambda_1,\lambda_2]$. With the action of the grading operator d on the highest weight state fixed by $d\,|\kappa\rangle=0$, the eigenvalues of d on any weight state are called its grade. According to (103) and the defining relations (1)-(5), the following weights are found in $V(\Lambda_2)$ at grade 0 and -1:

$$\begin{array}{ccc} & & \\ [0,0,1]_1 & & \\ & \searrow & \\ & & [1,-1,1]_1 \\ & \swarrow & \\ [2,-1,0]_1 & & \end{array} \qquad\qquad \begin{array}{ccccccc} & & & [-1,0,2]_1 & & & \\ & & \swarrow & & \searrow & & \\ & [0,0,1]_2 & & & & [0,-1,2]_1 & \\ \swarrow & & \searrow & & \swarrow & & \\ [1,0,0]_1 & & & [1,-1,1]_3 & & & \\ \searrow & & \swarrow & & \searrow & & \\ & [2,-1,0]_2 & & & & [2,-2,1]_1 & \\ & & \searrow & & \swarrow & & \\ & & & [3,-2,0]_1 & & & \end{array} \tag{104}$$

Here the left and right part of the diagram refers to the weight space at grade 0 and -1, respectively, and the subscripts denote the multiplicity of the weights. Arrows pointing southwest (southeast) indicate the action of f_1 (f_2). Identifying the grade of a weight state with the value $\bar{H}_0$ associated to the configurations listed above, a one-to-one correspondence between the pairs λ_1,λ_2 and the values $\bar{h}_1,\bar{h}_2$ is found at grade 0 and -1. This correspondence applies to the next three lower grades, too.

The states of $V(\Lambda_2)$ at a fixed nonvanishing grade can be arranged arranged in a finite number of separate sets labeled by pairs (λ_1,λ_2) with $\lambda_1\geq 0$ and $\lambda_2>0$. A set labeled by will be called $\pi(\lambda_1,\lambda_2)$ in the following. It contains $4(\lambda_1+\lambda_2)$ states with weights$(1-\lambda_1-\lambda_2,\lambda_1,\lambda_2)$, $(1+2\mu-\lambda_1-\lambda_2,\lambda_1-\mu,\lambda_2-\mu)$, $(2\mu-\lambda_1-\lambda_2,\lambda_1-\mu+1,\lambda_2-\mu)$ and $(2\mu-\lambda_1,-\lambda_2,\lambda_1-\mu,\lambda_2-\mu+1)$ with $0<\mu\leq\lambda_1+\lambda_2$. The multiplicity is two for each weight $(1+2\mu-\lambda_1-\lambda_2,\lambda_1-\mu,\lambda_2-\mu)$ with $0<\mu<\lambda_1+\lambda_2$ and one for all others. For the four lowest nonvanishing grades, the pairs (λ_1,λ_2) are listed in the following table with the multiplicity specified by a subscript:

grade -1	grade -2	grade -3	grade -4
$[0,2]_1$	$[1,2]_1$	$[1,2]_2$	$[1,3]_1$
$[0,1]_1$	$[0,2]_2$	$[0,3]_1$	$[1,2]_5$
	$[0,1]_2$	$[1,1]_1$	$[0,3]_2$
	$[-1,2]_1$	$[0,2]_5$	$[1,1]_3$
		$[0,1]_4$	$[0,2]_{10}$
		$[-1,2]_2$	$[0,1]_8$
			$[-1,2]_5$
			$[-1,3]_1$

(105)

Collecting all configurations with $\bar{H}_0$ given by $-2,-3,-4$ reveals that the number of configurations with values $(\bar{H}_0, \bar{h}_1, \bar{h}_2)$ coincides with the multiplicity of the weight $(1-\bar{h}_1-\bar{h}_2, \bar{h}_1, \bar{h}_2)$ at grade $\bar{H}_0$. Thus the one-to-one correspondence between CTM-configurations and weight states of the module $V(\lambda_2)$ at level one is confirmed down to grade -4.

Equation (101) is involved in the evaluation for the diagonal element of H_0 on the configurations $(\ldots, 2^*2, 2^*2, 2\circ 0^*)$ and $(\ldots, 2^*2, 2^*2, 2\circ 1^*)$ at grade -2, on $(\ldots, 2^*2, 2^*2, 2\circ 2^*)$, $(\ldots, 2^*2, 2^*2\{\circ-(+-)\}j^*)$, $(\ldots, 2^*2, 2^*2, 2^*i\circ j^*)$, $(\ldots, 2^*2, 2^*1, 0\circ j^*)$, $(\ldots, 2^*2, 2^*2, +\circ j^*)$, $(\ldots, 2^*2, 2^*2, -\circ j^*)$ and $(\ldots, 2^*2, 2^*2\circ 1^*, 2j^*)$ with $i, j = 0, 1$ at grade -3 and on 57 configurations at grade -4.

Due to (56), a configuration $(\ldots, 2^*2, 2^*2, j^*_{n+1}i_n, j^*_n i_{n-1}, \ldots, j^*_3 i_2, j^*_2 i_1 j^*_1)$ with $j^*_{n+1}\neq 2$ or $j^*_{n+1}=2, i_n\neq 2$ not containing any subsequence (57) has the diagonal element of H_0 bound by $-n(n+1)\leq \bar{H}_0$. In particular, the value of the lower bound is taken for the configuration $\big(\ldots, 2^*2, 2^*2, 2^*(0,2^*)^n\big)$. As is easily verified from (56) and (101), all other configurations with the values $(\bar{H}_0, n, n+1)$ satisfy $\bar{H}_0 < -n(n+1)$. The corresponding state in $V(\Lambda_2)$ is $E^{2,+}_{-n}\ldots E^{2,+}_{-2}E^{2,+}_{-1}\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle$ with weight $(-2n, n, n+1)$ and grade $-n(n+1)$. At level one, $f_0\,|\kappa\rangle = 0$. Because of this property and (11), any other state in $V(\Lambda_2)$ with the same weight has a lower grade.

Generally, the weights $(-m-n, m, n+1)$ and $(2+m+n, -n-1, -m)$ with $n\geq m\geq 0$ appear in the module $V(\Lambda_2)$ at grades bounded from above by $-\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$. At grade $\leq -\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$ the multiplicity is one. In terms of Drinfeld generators, the corresponding states read

$$E^{2,+}_{-m}\ldots E^{2,+}_{-2}E^{2,+}_{-1}\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle \qquad \text{and} \qquad (f_1 f_2)^{m+n+1}\, E^{2,+}_{-m}\ldots E^{2,+}_{-2}E^{2,+}_{-1}\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle \tag{106}$$

Besides, the weights $(1-n, -1, n+1)$ and $(1+n, -n-1, 1)$ are found at grade $-\frac{1}{2}n(n+1)$ with multiplicity one. The corresponding states $E^{2,-}_0\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle$ and $E^{2,-}_0\,(E^{1,-}_0 E^{2,-}_0)^n\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle$ belong to the set $\pi(0, n+1)$ at this grade. For the grade $-\frac{1}{2}n(n+1)-1$, the multiplicities of the weights $(-n-1, 1, n+1)$, $(-n, 0, n+1)$, $(1-n, -1, n+1)$ and $(2-n, -2, n+1)$ are 1, 3, 3 and 1, respectively. Hence, the last two weights are contained in a set $\pi(-1, n+1)$ with the weight $(1-n, -1, n+1)$ related to the state $E^{2,-}_{-1}\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle$. Similarly, sets $\pi(-m, n+1)$ with $n\geq m>0$ are present at any grade $\leq -\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$. Exactly one set is found at grade $-\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$. The state corresponding to its weights are generated by acting with f_1 and f_2 on the states

$$E^{2,-}_{-m}\ldots E^{2,-}_{-2}E^{2,-}_{-1}\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle \qquad n\geq m>0 \tag{107}$$

For $n\geq m\geq 0$, the weights $(1+2k+m-n, -m-k-1, n+1-k)$ with $0\leq k\leq n+m$ contained in the sets labeled by $(-m, n+1)$ have multiplicity one at the maximal grade.

In addition, sets $\pi(n+m+1, n+1)$ with $m, n\geq 0$ appear at all grades $\leq -(n+1)(n+m+1)-\frac{1}{2}(m+1)(m+2)-\delta_{n,0}$. Again, there is exactly one set at the maximal grade. The states related to its weights arise from the action of f_1 and f_2 on

$$\begin{aligned}
&E^{1,-}_{-1}\, E^{2,+}_{-1}E^{1,+}_{-1}\,|\kappa\rangle && n=m=0\\
&E^{1,-}_{-1}\, E^{2,+}_{-(m+2)}\ldots E^{2,+}_{-4}E^{2,+}_{-3}\, E^{2,+}_{-1}E^{1,+}_{-1}\,|\kappa\rangle && n=0,\ m>0\\
&E^{2,+}_{-(n+m+2)}\ldots E^{2,+}_{-(n+3)}E^{2,+}_{-(n+2)}\, E^{2,+}_{-n}\ldots E^{2,+}_{-2}E^{2,+}_{-1}\, E^{1,+}_{-n}\ldots E^{1,+}_{-2}E^{1,+}_{-1}\,|\kappa\rangle && n>0,\ m\geq 0
\end{aligned} \tag{108}$$

At the maximal grade $-(n+1)(n+m+1)-\frac{1}{2}(m+1)(m+2)-\delta_{n,0}$, the weights $(n+m-k, n-k)$ with $n>0,\ m\geq 0$ and $0\leq k\leq 2n+m$ found in the sets $(n+m, n+1)$ have multiplicity one. The complete collection of all states with multiplicity one present in the level-one module $V(\Lambda_2)$ is obtained by adding the zero-grade states $|\kappa\rangle$ and $E^{1,-}_0 E^{2,-}_0\,|\kappa\rangle$ to those listed so fare.

The weights belonging to the sets $\pi(m,n)$ for $m\geq 0$ and $n>0$ exhaust all weights in the module $V(\Lambda_2)$ at level one. The upper bounds for the grades with a set $\pi(m,n)$ with $n>m\geq 0$ present should be compared with the maximal diagonal element of H_0 for the configurations with eigenvalues $\bar{h}_1=m$, $\bar{h}_2=n$ and $\bar{h}_1=-n$, $\bar{h}_2=-m$. From (56) and (101), these are the configurations

$$\big(\ldots, 2^*2, 2^*2, 2^*(0,1^*)^{n-m}(0,2^*)^m\big) \qquad \text{and} \qquad \big(\ldots, 2^*2, 2^*2, 2^*(2,1^*)^{n-m}(2,0^*)^{m+1}\big) \qquad n>m\geq 0 \tag{109}$$

with $\bar{H}_0 = -\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$. There are no other configurations with the same values $(\bar{H}_0, \bar{h}_1, \bar{h}_2)$. Therefore the configurations (109) can be viewed as counterparts of the states (106).

Moreover, configurations with $\bar{h}_1 = -m-1-k$, $\bar{h}_2 = n+1-k$ with $0\leq k\leq n+m$ are found only with $\bar{H}_0\leq -\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$. The maximal value of $\bar{H}_0$ is attributed to the configurations

$$\big(\ldots,2^*2,2^*2,1^*(0,1^*2,1^*)^m(2,1^*)^k(0,1^*)^{n-m-k}\big) \qquad n\ge m\ge 0,\ 0\le k\le n+m \tag{110}$$

Thus the configurations (110) can be related to the states $E_0^{2,-}(E_0^{1,-}E_0^{2,-})^k\ E_{-n}^{1,+}\ldots E_{-2}^{1,+}E_{-1}^{1,+}\,|\kappa\rangle$ for $m=0$ and to $E_0^{2,-}\,(E_0^{1,-}E_0^{2,-})^kE_{-m}^{2,-}\ldots E_{-2}^{2,-}E_{-1}^{2,-}\ E_{-n}^{1,+}\ldots E_{-2}^{1,+}E_{-1}^{1,+}\,|\kappa\rangle$ for $m>0$. The weights of these states are part of the sets $\pi(-m,n+1)$ at grade $-\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)$ with $n\ge m\ge 0$.

Configurations with $\bar h_1=m+1-k$ and $\bar h_2=-k$ with $m\ge 0$ and $0\le k\le m+1$ exist with $\bar H_0\le -\frac{1}{2}(m+2)(m+3)$ with the maximal value of $\bar H_0$ valid for the configurations

$$\big(\ldots,2^*2,2^*2,2^*(1,0^*)^{k+1}(1,2^*)^{m+1-k}\big) \qquad m\ge 0,\ 0\le k\le m+1 \tag{111}$$

Finally, configurations with $\bar h_1=n+m-k$ and $\bar h_2=n-k$ with $n>0,m\ge 0$ and $0\le k\le m+2n$ are present only with $\bar H_0\le -(n+m)(n+1)-\frac{1}{2}m(m+1)$. In the case $n=1$, the maximal value of $\bar H_0$ is attributed to the configurations

$$\begin{aligned}
&\big(\ldots,2^*2,2^*2,2^*(1,2^*)^{m+1}\big) && \text{for } k=0\\
&\big(\ldots,2^*2,2^*2,0^*(1,0^*)^{k-1}(1,2^*)^{m+2-k}\big) && \text{for } 1\le k\le m+1\\
&\big(\ldots,2^*2,2^*2,2^*(1,0^*)^{m+1}\big) && \text{for } k=m+2
\end{aligned} \tag{112}$$

with $m\ge 0$. If $n>1$, the maximal value of $\bar H_0$ applies to the configurations

$$\begin{aligned}
&\big(\ldots,2^*2,2^*2,2^*(1,2^*)^{m+1}(0,2^*)^{n-1}\big) && \text{for } k=0\\
&\big(\ldots,2^*2,2^*2,0^*(1,0^*)^{k-1}(1,2^*)^{m+2-k}(0,2^*)^{n-1}\big) && \text{for } 1\le k\le m+1\\
&\big(\ldots,2^*2,2^*2,0^*(1,0^*)^{m}(2,0^*)^{l}2,2^*1,2^*(0,2^*)^{n-l-2}\big) && \text{for } k=m+2l+2,\ 0\le l\le n-2\\
&\big(\ldots,2^*2,2^*2,0^*(1,0^*)^{m}(2,0^*)^{l+1}1,2^*(0,2^*)^{n-l-2}\big) && \text{for } k=m+2l+3,\ 0\le l\le n-2\\
&\big(\ldots,2^*2,2^*2,0^*(1,0^*)^{m}(2,0^*)^{n-1}1,0^*\big) && \text{for } k=m+2n
\end{aligned} \tag{113}$$

with $m\ge 0$. The configurations (111), (112) and (113) are counterparts of the states with weights $(1+2k-2n-m,n+m-k,n-k)$ in the set labeled by $(m+n,n+1)$ at grade $-(n+m)(n+1)-\frac{1}{2}m(m+1)$ for any $n,m\ge 0$, $n+m>0$.

As is easily verified from (56) and (101), the values $(\bar H_0,\bar h_1,\bar h_2)$ for a configuration with $\bar H_0$ determined by (101) are shared by at least two different configurations. If (56) covers the evaluation of $\bar H_0$ for a configuration containing a subsequence (1^*1+0^*0), the value of $\bar H_0$ does not change when replacing this subsequence by (1^*1-0^*0). The values $\bar h_1$, $\bar h_2$ for any other configuration with $\bar H_0$ determined by (56) can be attributed to a weight in one of the sets $\pi(m,n)$ related to the configurations (109)-(113) as specified above. Again the grade coincides with the value $\bar H_0$. For example, $\big(\ldots,2^*2,2^*(2,0^*)^k2,1^*(0,2^*)^{n-k}\big)$ with $0\le k\le n$ and $\big(\ldots,2^*2,2^*(2,0^*)^k2,1^*2,2^*(0,2^*)^{n-k-2}\big)$ with $0\le k\le n-1$ correspond to the weights $(1+4k-2n,n-2k-1,n-2k+1)$ and $(3+4k-2n,n-2k-2,n-2k)$ of the set $\pi(n,n+1)$ at grade $-n(n+1)$, $n>1$. Due to the weight structure of the sets, any values $(\bar H_0,\bar h_1,\bar h_2)$ different from $\big(-\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1),m,n+1\big)$, $\big(-\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1),-n-1,-m\big)$ with $n\ge m\ge 0$ or $\big(-\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1),-m-k-1,n+1-k\big)$ with $n\ge m\ge 0$, $0\le k\le n+m$ or $\big(-(n+m)(n+1)-\frac{1}{2}m(m+1)-\delta_{n,0},n+m-k,n-k\big)$ with $n,m\ge 0$, $n+m>0$ and $0\le k\le 2n+m$ or $(0,-k,1-k)$ with $k=0,1$ occur more than once.

Thus each state contained in the irreducible level-one module $V(\Lambda_2)$ with multiplicity can be mapped onto exactly one of the CTM-configurations with a nondegenerated triple $(\bar H_0,\bar h_1,\bar h_2)$ and vice versa. This observation as well as the result on all states at grade 0 to -4 support the conjecture that the CTM-configurations $(\ldots,j_3^*i_2,j_2^*i_1,j_1^*)$ and the weight states of $V(\Lambda_2)$ at level one are in one-to-one correspondence.

VII. APPENDIX A

All subsequences generated by the action of $h^{(n,\kappa)}$ on $\big((-)^{M-K}(\circ)^K\big)$ for $M\ge K$ or from $\big((+)^{K-M}(\circ)^M\big)$ for $K\ge M$ have to be expressible as linear combinations of the same terms and subsequences contained in the groups specified

by (83)-(93), if $\mathcal{K}(K,M) = \big\{\mathcal{K}'(K,M), \big((-)^{M-K}(\circ)^K\big)\big\}$ for $M \geq K$ or $\mathcal{K}(K,M) = \big\{\mathcal{K}'(K,M), \big((+)^{K-M}(\circ)^M\big)\big\}$ for $K \geq M$ is to provide a basis of all subsequences of the form (67) with the same values of K and M.

It is convenient to consider the case $K = M$ first. The relevant subsequences are then given by $\big((\circ)^{M-1-L}\big\{\circ - (\rho, -\rho)\big\}(\circ)^L\big)$ with $\rho = \pm$ and $0 \leq L \leq M-1$. For $K = M = 3$, the required rewritings read

$$\big(\big\{\circ-(+-)\big\}\circ\circ\big) = -\big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}^2\big) + \big(\big\{\circ-(+-)\big\}\circ\big\{\circ-(-+)\big\}\big) + \big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}\circ\big)$$
$$-\big(\big\{\circ-(+-)\big\}-\big\{\circ-(+-)\big\}+\big) + \big(\big\{\circ-(+-)\big\}-\circ+\big) \tag{114}$$

and

$$\big(\circ\circ\big\{\circ-(+-)\big\}\big) = -\big(\big\{\circ-(-+)\big\}^2\big\{\circ-(+-)\big\}\big) + \big(\big\{\circ-(-+)\big\}\circ\big\{\circ-(+-)\big\}\big) + \big(\circ\big\{\circ-(-+)\big\}\big\{\circ-(+-)\big\}$$
$$-\big(-\big\{\circ-(+-)\big\}+\big\{\circ-(+-)\big\}\big) + \big(-\circ+\big\{\circ-(+-)\big\}\big) \tag{115}$$

Exchanging $+ \leftrightarrow -$ in (114), (115) yields $\big(\big\{\circ-(-+)\big\}\circ\circ\big)$ and $\big(\circ\circ\big\{\circ-(-+)\big\}\big)$ in terms of subsequences of $\mathcal{K}'(3,3)$. These expressions and (114), (115) may be combined to rewrite a subsequence built from several terms $(\circ)^k$ with $k = 1,2$ separated by $\big\{\circ-(-+)\big\}$ and/or $\big\{\circ-(+-)\big\}$ in terms of subsequences contained in $\mathcal{K}'(M,M)$. The equations

$$\big(a\big\{\circ-(-+)\big\}(+)^s\circ(-)^r\big) = \big(a\big\{\circ-(-+)\big\}(+)^{s+1}\circ(-)^{r-1}\big) + \big(a\big\{\circ-(-+)\big\}(+)^s\big\{\circ-(+-)\big\}(+)^r\big)$$
$$-\big(a\big\{\circ-(-+)\big\}(+)^{s+1}\big\{\circ-(-+)\big\}(+)^{r-1}\big) \tag{116}$$

with $s \geq 0$, $r \geq 1$ and a part a as described below (75) allow to reformulate $\big(\big\{\circ-(+-)\big\}\circ\circ(+)^r\big)$ in terms of subsequences found in $\mathcal{K}'(r+3,3)$:

$$\big(\big\{\circ-(+-)\big\}\circ\circ(+)^r\big) = -\big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}^2\big) + \big(\big\{\circ-(+-)\big\}\circ\big\{\circ-(+-)\big\}(+)^r\big)$$
$$+\big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}(+)^r\circ\big) + \sum_{t=0}^{r-1}\big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}(+)^t\big\{\circ-(+-)\big\}(+)^{r-t}\big)$$
$$-\sum_{t=1}^{r}\big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}(+)^t\big\{\circ-(-+)\big\}(+)^{r-t}\big)$$
$$-\big(\big\{\circ-(+-)\big\}-\big\{\circ-(+-)\big\}(+)^{r+1}\big) + \big(\big\{\circ-(+-)\big\}-\circ(+)^{r+1}\big) \tag{117}$$

Furthermore, $\big\{\circ-(+-)\big\}$ in (114) and the leftmost term $\big\{\circ-(+-)\big\}$ in (117) can be substituted by $\big\{\circ-(+-)\big\}(-)^s$ with $s = 1,2,3,\ldots$. Reversing the order of symbols and exchanging $+ \leftrightarrow -$ provides expressions for $\big((-)^r\circ\circ(+)^s\big\{\circ-(+-)\big\}\big)$, $\big(\big\{\circ-(-+)\big\}(+)^s\circ\circ(-)^r\big)$ and $\big((+)^r\circ\circ(-)^s\big\{\circ-(-+)\big\}\big)$. Equation (114) may be generalized starting from

$$\Big(\big\{\circ-(+-)\big\}(\circ)^{M-1}\Big) + (-1)^n\Big(\big\{\circ-(+-)\big\}(-+)^{M-1}\Big) =$$
$$\Big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}^{M-1}\Big)+$$
$$+\sum_{r=1}^{M-2}(-1)^r\sum_{k_1=0}^{M-r-1}\;\sum_{k_2=0}^{M-r-1-k_1}\;\sum_{k_3=0}^{M-r-1-k_1-k_2}\cdots\sum_{k_r=0}^{M-r-1-k_1-\ldots-k_{r-1}}$$
$$\Big(\big\{\circ-(+-)\big\}\big\{\circ-(-+)\big\}^{k_1}\circ\big\{\circ-(-+)\big\}^{k_2}\circ\ldots\circ\big\{\circ-(-+)\big\}^{k_r}\circ\big\{\circ-(-+)\big\}^{M-r-1-k_1-\ldots-k_r}\Big) \tag{118}$$

Again, the leftmost term $\big\{\circ-(+-)\big\}$ may be replaced by $\big\{\circ-(+-)\big\}-$. In order to obtain an expression for $\big(\big\{\circ-(+-)\big\}(\circ)^{M-1}+\big)$ or $\big(\big\{\circ-(+-)\big\}-(\circ)^{M-1}+\big)$ by inserting a symbol $+$ left of the right border of (118), an appropriate rewriting of $\big(a\big\{\circ-(-+)\big\}(\circ)^n+\big)$ is required:

$$\Big(a\big\{\circ-(-+)\big\}(\circ)^n+\Big)+(-1)^n\Big(a\big\{\circ-(-+)\big\}(+-)^n+\Big)=$$

$$=\Big(a\big\{\circ-(-+)\big\}\big\{\circ-(+-)\big\}^n+\Big)+$$

$$+\sum_{r=1}^{n-1}(-1)^r\sum_{k_1=0}^{n-r}\quad\sum_{k_2=0}^{n-r-k_1}\quad\sum_{k_3=0}^{n-r-k_1-k_2}\cdots\sum_{k_r=0}^{n-r-k_1-k_2-\ldots-k_{r-1}}$$

$$\Big(a\big\{\circ-(-+)\big\}\big\{\circ-(+-)\big\}^{k_1}\circ\big\{\circ-(+-)\big\}^{k_2}\circ\ldots\circ\big\{\circ-(+-)\big\}^{k_r}\circ\big\{\circ-(+-)\big\}^{n-r-k_1-\ldots-k_r}+\Big) \tag{119}$$

On the other hand,

$$\Big(a\big\{\circ-(-+)\big\}+(\circ)^n\Big)+(-1)^n\Big(a\big\{\circ-(-+)\big\}+(-+)^n\Big)=$$

$$=\Big(a\big\{\circ-(-+)\big\}+\big\{\circ-(-+)\big\}^n\Big)+$$

$$+\sum_{r=1}^{n-1}(-1)^r\sum_{k_1=0}^{n-r}\quad\sum_{k_2=0}^{n-r-k_1}\quad\sum_{k_3=0}^{n-r-k_1-k_2}\cdots\sum_{k_r=0}^{n-r-k_1-k_2-\ldots-k_{r-1}}$$

$$\Big(a\big\{\circ-(-+)\big\}+\big\{\circ-(-+)\big\}^{k_1}\circ\big\{\circ-(-+)\big\}^{k_2}\circ\ldots\circ\big\{\circ-(-+)\big\}^{k_r}\circ\big\{\circ-(-+)\big\}^{n-r-k_1-\ldots-k_r}\Big) \tag{120}$$

A reformulation for the first term on the lhs of (120) is provided by replacing the left most term $\big\{\circ-(+-)\big\}$ in (118) by $\big\{\circ-(+-)\big\}-$, switching all signs and adding the piece a at the left border. Combining this with equations (119) and (120) leads to an expression of $\big(\{\circ-(+-)\}-(\circ)^{M-2}+\big)$ analogous to the equation (118) for $\big(\{\circ-(+-)\}(\circ)^{M-1}\big)$. Use of this expression in

$$\Big(\big\{\circ-(+-)\big\}-(\circ)^{M-2}+\Big)-(-1)^n\Big(\big\{\circ-(+-)\big\}(-+)^{M-1}\Big)=$$

$$\Big(\big\{\circ-(+-)\big\}-\big\{\circ-(+-)\big\}^{M-2}+\Big)$$

$$-\sum_{r=1}^{M-3}(-)^r\sum_{k_1=0}^{M-r-2}\quad\sum_{k_2=0}^{M-r-2-k_1}\quad\sum_{k_3=0}^{M-r-2-k_1-k_2}\cdots\sum_{k_r=0}^{M-r-2-k_1-\ldots-k_{r-1}}$$

$$\Big(\big\{\circ-(+-)\big\}-\big\{\circ-(+-)\big\}^{k_1}\circ\big\{\circ-(+-)\big\}^{k_2}\circ\ldots\circ\big\{\circ-(+-)\big\}^{k_r}\big\{\circ-(+-)\big\}^{M-r-2-k_1-\ldots-k_r}\Big) \tag{121}$$

and comparison with (118) yields an rewriting of $\big(\{\circ-(+-)\}(\circ)^{M-1}\big)$ in terms of subsequences involving lower powers of $\circ$. Analogous expressions for $\big((\circ)^{M-1}\{\circ-(+-)\}\big)$, $\big(\{\circ-(-+)\}(\circ)^{M-1}\big)$ and $\big((\circ)^{M-1}\{\circ-(-+)\}\big)$ are obtained by reversing the order of symbols and/or exchanging $+\leftrightarrow-$. These may be combined to provide reformulations of subsequences containing powers $(\circ)^m$ with $m\le M-1$ separated by $\{\circ-(-+)\}$ and/or $\{\circ-(+-)\}$. Together with (114)-(117), this ensures the existence of rewritings of $\big(a\{\circ-(\rho,-\rho)\}(\circ)^n\big)$ and $\big((\circ)^n\{\circ-(\rho,-\rho)\}a\big)$ in terms of subsequences of $\mathcal{K}'(K,M)$ with K,M according to the piece a. Hence all subsequences $\big((\circ)^{M-1-L}\{\circ-(\rho,-\rho)\}(\circ)^L\big)$ can be reformulated as linear combinations of subsequences found in $\mathcal{K}(M,M)$. Continuing along this lines it is straightforward to give such rewritings for any subsequence of the form (67) with $K=M$.

For $M>K$, all subsequences $\big((-)^{M-K-t_0-t_1-\ldots-t_K}\circ(-)^{t_K}\circ\ldots(-)^{t_2}\circ(-)^{t_1}\circ(-)^{t_0}\big)$ with $t_l\ge0\ \forall l$ and $M-K-\sum_{l=0}^K t_l\ge0$ can be rewritten in terms of $\big((-)^{M-K}(\circ)^K\big)$ and subsequences of $\mathcal{K}'(K,M)$. This is easily demonstrated in the case $K=1$:

$$\big((-)^{M-1-L}\circ(-)^L\big)-\big((-)^{M-2-L}\circ(-)^{L+1}\big)=$$

$$=\big((-)^{M-1-L}\{\circ-(+-)\}(-)^L\big)-\big((-)^{M-2-L}\{\circ-(-+)\}(-)^{L+1}\big) \tag{122}$$

for $0 \leq L \leq M-2$. Hence all subsequences can be obtained by adding $\big((-)^{M-1}\circ\big)$ to the collection $\mathcal{K}'(1,M)$. Of course, any other $\big((-)^{M-1-L}\circ(-)^{L}\big)$ would be appropriate as well for supplementing $\mathcal{K}'(1,M)$. Similarly, for $K=2$,

$$\begin{aligned}&\big((-)^{M-3-L}\circ(-)^{L+1}\circ\big)-\big((-)^{M-2-L}\circ(-)^{L}\circ\big)=\\&\qquad=\big((-)^{M-3-L}\{\circ-(-+)\}(-)^{L+1}\circ\big)-\big((-)^{M-2-L}\{\circ-(+-)\}(-)^{L}\circ\big)\end{aligned}\tag{123}$$

for $0 \leq L \leq M-3$. The rhs of (123) can be expressed in terms of subsequences of $\mathcal{K}'(2,M)$ using

$$\begin{aligned}&\big((-)^{M-2-L}\{\circ-(-+)\}(-)^{L-L'}\circ(-)^{L'}\big)=\big((-)^{M-2-L}\{\circ-(-+)\}(-)^{L-L'-1}\circ(-)^{L'+1}\big)+\\&+\big((-)^{M-2-L}\{\circ-(-+)\}(-)^{L-L'}\{\circ-(+-)\}(-)^{L'}\big)-\big((-)^{M-2-L}\{\circ-(-+)\}(-)^{L-L'+1}\{\circ-(-+)\}(-)^{L'+1}\big)\end{aligned}\tag{124}$$

with $0 \leq L \leq M-2$ and $0 \leq L' \leq L$. The order of symbols may be reversed. Thus, any $\big((-)^{M-2-L}\circ(-)^{L}\circ\big)$ or $\big(\circ(-)^{L}\circ(-)^{M-2-L}\big)$ can be expressed by $\big((-)^{M-2}\circ\circ\big)$ and $\mathcal{K}'(2,M)$. Moreover, the rhs of

$$\begin{aligned}&\big((-)^{M-3-L-L'}\circ(-)^{L+1}\circ(-)^{L'}\big)-\big((-)^{M-2-L-L'}\circ(-)^{L}\circ(-)^{L'}\big)=\\&\qquad=\big((-)^{M-3-L-L'}\{\circ-(-+)\}(-)^{L+1}\circ(-)^{L}\big)-\big((-)^{M-2-L-L'}\{\circ-(+-)\}(-)^{L}\circ(-)^{L'}\big)\end{aligned}\tag{125}$$

with $L' \geq 1$ and $0 \leq L \leq M-3-L'$ can be rewritten in terms of the same set of subsequences by repeated use of

$$\begin{aligned}&\big((-)^{M-2-L}\{\circ-(+-)\}(-)^{L-L'}\circ(-)^{L'}\big)=\big((-)^{M-2-L}\{\circ-(+-)\}(-)^{L-L'+1}\circ(-)^{L'-1}\big)+\\&+\big((-)^{M-2-L}\{\circ-(+-)\}(-)^{L-L'}\{\circ-(-+)\}(-)^{L'}\big)-\big((-)^{M-2-L}\{\circ-(+-)\}(-)^{L-L'+1}\{\circ-(+-)\}(-)^{L'-1}\big)\end{aligned}\tag{126}$$

for $0 \leq L \leq M-2$ and $1 \leq L' \leq L$. The order of symbols may be reversed in (125) and (126). This accounts for the required rewriting of any subsequence $\big((-)^{M-2-L-L'}\circ(-)^{L}\circ(-)^{L'}\big)$ with $0 \leq L \leq M-2$ and $0 \leq L' \leq M-2-L$. Continuing with this procedure the statement is readily extended to all subsequences of the form (67) with $K=2$. Upon repeated application of

$$\big(b-\circ a\big)-\big(b\circ -a\big)=\big(b-\{\circ-(+-)\}a\big)-\big(b\{\circ-(-+)\}-a\big)\tag{127}$$

the above procedure for $K=2$ generalizes in a straightforward manner to general $K<M$. Switching all signs and exchanging $M \leftrightarrow K$ yields the corresponding formulae for the case $K>M$.

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